A Gentle Introduction to the Circle Method

Daniel Flores

Purdue University

flore205@purdue.edu

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Overview

Beginnings: The Partition Function

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What We Know

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Next Question

What about an asymptotic formula?

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Theorem (Hardy, Ramanujan, *Asymptotic formulae in combinatory analysis*, 1918)

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Here we say that $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.
Their proof was based on the observation that one the following expression for the generating function

$$f(z) = \sum_{n=0}^{\infty} P(n) z^n = \prod_{k=1}^{\infty} \left(\frac{1}{1 - z^k} \right),$$

thus one may apply Cauchy's integral formula to obtain

$$P(n) = \frac{1}{2\pi i} \oint \frac{f(z)}{z^{n+1}} dz.$$

Beginnings: Waring's Problem

Definition

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One has $g(2) = 4$, $g(3) = 9$, $g(4) = 19, \ldots$, and for each $k \in \mathbb{N}$ one has $g(k) < \infty$.

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Theorem (Hilbert, 1909)

For each $k \in \mathbb{N}$ one has $g(k) < \infty$.

The exact formula for $g(k)$ is known to be

$$2^k + \left\lfloor (3/2)^k \right\rfloor - 2$$

for all but a finite (possibly empty) set of k. The reason for this is because the representation of small n as a sum of k-th powers requires an abnormally large number of variables.

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Alternate Definition

Given $k \in \mathbb{N}$, we define $G(k)$ to be the smallest integer s having the property that all <u>sufficiently large enough</u> natural numbers are the sum of at most s positive integral k-th powers.

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World Record Results!

$G(2) = 4$	Lagrange (1770)
$G(3) \leqslant 7$	Linnik (1942)
$G(4) = 16$	Davenport (1939)
$G(5) \leqslant 17$	Vaughan, Wooley (1995)
$G(6) \leqslant 24$	Vaughan, Wooley (1994)
$G(7) \leqslant 31$	Wooley (2016)
$G(8) \leqslant 39$	Wooley (2016)
$G(9) \leqslant 47$	Wooley (2016)
$G(k) \leqslant \lceil k(\log k + 4.20032) \rceil$	Brüdern, Wooley (2023)

Overview of the Circle Method

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$$r(n) = \int_0^1 S(\alpha)e(-\alpha n)d\alpha.$$

Typically $r(n) = 0$ for all n greater than some parameter X, or $r(n)$ rapidly decays to 0 for $n > X$.

Overview of the Circle Method

For example if we define

$$S(\alpha) = \left(\sum_{1 \leqslant x \leqslant X} e(\alpha x^k) \right)^s,$$

then then one can easily show that

$$r(n) = r(n; s, k, X) = \#\{x_1, \dots, x_s \in [1, X] \cap \mathbb{N} : x_1^k + \cdots + x_s^k = n\}.$$

If one can show $r(n; s_0, k_0, \lfloor n^{1/k} \rfloor) \geqslant 1$ for a fixed pair of (s_0, k_0) and all sufficiently large n then we may conclude that $G(k_0) \leqslant s_0$.

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Important Insight of Hardy-Littlewood

They noticed that the modulus of the generating function $S(\alpha)$ was quite large when $\alpha = a/q$ and $q = o(X)$. In the opposite direction, whenever α was not well approximable by rationals of low denominator then the modulus of the generating function $S(\alpha)$ should exhibit some cancellation and be small.

Major and Minor arc dissections

Thus were born the major arcs, which in the classical Waring's problem are taken to be the following.

Major Arcs

$$\mathfrak{M}_\delta = \bigcup_{\substack{0 \leqslant a \leqslant q \leqslant X^\delta \\ (a,q)=1}} \{\alpha \in [0,1) : |\alpha - a/q| \leqslant X^{\delta - k}\}$$

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Minor Arcs

$$\mathfrak{m}_\delta = [0,1) \backslash \mathfrak{M}_\delta$$

Note: From now on we will be taking $X = \lceil n^{1/k} \rceil$.

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$$I_2 := \int_{\mathfrak{m}_\delta} S(\alpha)e(-\alpha n)d\alpha,$$

should exhibit some cancellation and therefore have smaller rate of growth than the main term.

Dealing with the Minor arcs

Define $f_k(\alpha) = \sum_{1 \leqslant x \leqslant X} e(\alpha x^k)$. Now we cite some important results in the field of exponential sums.

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By Weyl's Inequality

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By Hua's Lemma

$$\int_0^1 |f_k(\alpha)|^{2^k} d\alpha \ll X^{2^k-k+\epsilon}$$

Dealing with the Minor arcs

Returning to our definition of I_2 and citing the previous results, one has by Hölder's inequality that whenever $s \geqslant 2^k + 1$ we obtain

$$\begin{aligned} I_2 &\leqslant \sup_{\alpha \in \mathfrak{m}_\delta} |f(\alpha)|^{s-2^k} \int_0^1 |f_k(\alpha)|^{2^k} d\alpha \\ &\ll X^{(s-2^k)(1-\delta 2^{1-k}+\epsilon)+2^k-k+\epsilon} \\ &= o(n^{s/k-1}) \end{aligned}$$

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Which is good enough, because (spoiler) the major arc contribution grows like a positive multiple of $n^{s/k-1}$.

Dealing with the Major arcs

When α is well approximated by the rational number a/q we expect that

$$f_k(\alpha) \sim q^{-1} \left(\sum_{1 \leqslant r \leqslant q} e(ar^k/q) \right) \left(\int_0^X e(\beta\gamma^k) d\gamma \right).$$

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Thus after some work one may arrive at the conclusion

$$I_1 \sim \sigma_\infty \left(\prod_{p \text{ prime}} \sigma(p) \right) n^{s/k-1}.$$

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$$\sigma_\infty = \lim_{\eta \to 0^+} \eta^{-1} \text{mes} \left\{ \mathbf{x} \in [0,1]^s : |x_1^k + \cdots + x_s^k - n/X^k| < \eta \right\},$$

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and hence may be regarded as a "real" density of solutions. Similarly, in a p-adic sense, one may show that

$$\sigma(p) = \lim_{h \to \infty} p^{h(1-s)} \#\{\mathbf{x} \in (\mathbb{Z}/p^h\mathbb{Z})^s : x_1^k + \cdots + x_s^k \equiv n \bmod p^h\},$$

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is a p-adic density of solutions. Some minor problems are left to be dealt with but this overview will suffice for now.

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Here are some number theoretical results which have been established via the circle method.

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All odd numbers greater than 5 are the sum at most three primes.

Theorem (Magyar, Stein, Wainger, 2002)

The discrete maximal spherical operator A^* is bounded in $\ell_p(\mathbb{Z}^d)$ to itself when $p > \frac{d}{d-2}$ and $d \geqslant 5$.

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Twin Prime Conjecture

It is possible to prove the twin prime conjecture if one were to obtain improved bounds over the minor arcs. (In fact something stronger would be shown).

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Goldbach Conjecture

If we could improve our understanding of the minor arcs this could be proven, as the main term seems to stem from the minor arcs!

Thank you for listening!