A Circle Method Approach to K-Multimagic Squares

Daniel Flores

Purdue University

Canadian Number Theory Association XVI

flore205@purdue.edu

June 14, 2024

What is a K-multimagic square

Definition

Given $K \geqslant 2$ we say a matrix $\mathbf{Z} \in \mathbb{Z}^{N \times N}$ is a K-multimagic square of order N or $\mathbf{MMS}(K,N)$ for short if the matrices

$$\mathbf{Z}^{\circ k} := (z_{i,j}^k)_{1 \leqslant i,j \leqslant N},$$

remain magic squares for $1 \leqslant k \leqslant K$.

What is a K-multimagic square

Definition

Given $K \geqslant 2$ we say a matrix $\mathbf{Z} \in \mathbb{Z}^{N \times N}$ is a K-multimagic square of order N or $\mathbf{MMS}(K, N)$ for short if the matrices

$$\mathbf{Z}^{\circ k} := (z_{i,j}^k)_{1 \leqslant i,j \leqslant N},$$

remain magic squares for $1 \leqslant k \leqslant K$.

Note: Here we do not require the elements to be distinct.

What is a K-multimagic square

Definition

Given $K \geqslant 2$ we say a matrix $\mathbf{Z} \in \mathbb{Z}^{N \times N}$ is a K-multimagic square of order N or $\mathbf{MMS}(K, N)$ for short if the matrices

$$\mathbf{Z}^{\circ k} := (z_{i,j}^k)_{1 \leqslant i,j \leqslant N},$$

remain magic squares for $1 \leqslant k \leqslant K$.

Note: Here we do not require the elements to be distinct.

Definition

If a $\mathbf{MMS}(K, N)$ contains every integer from 1 to N^2 then it is called a *normal* $\mathbf{MMS}(K, N)$.

Trivial Examples of $\mathbf{MMS}(K, N)$

1	1	1
1	1	1
1	1	1

This is a $\mathbf{MMS}(K, 3)$ for all $K \geqslant 2$.

Trivial Examples of $\mathbf{MMS}(K, N)$

1	1	1
1	1	1
1	1	1

This is a $\mathbf{MMS}(K, 3)$ for all $K \geqslant 2$.

a	b	c	d
d	c	b	a
b	a	d	c
c	d	a	b

For any $a, b, c, d \in \mathbb{Z}$ this is a $\mathbf{MMS}(K, 4)$ for all $K \geqslant 2$.

Definition of trivial $\mathbf{MMS}(K, N)$

Consideration of these "trivial" $\mathbf{MMS}(K, N)$ motivates the following definition.

Definition of trivial $\mathbf{MMS}(K, N)$

Consideration of these "trivial" $\mathbf{MMS}(K, N)$ motivates the following definition.

Definition

For $K \geqslant 2$ and $N \in \mathbb{N}$ a $\mathbf{MMS}(K, N)$ is called *trivial* if it utilizes N or less distinct integers.

Definition of trivial **MMS**(K, N)

Consideration of these "trivial" **MMS**(K, N) motivates the following definition.

Definition

For $K \geqslant 2$ and $N \in \mathbb{N}$ a **MMS**(K, N) is called *trivial* if it utilizes N or less distinct integers.

Question: Given $K \geqslant 2$ and $N \geqslant 4$ does there exists nontrivial **MMS**(K, N)?

Examples of nontrivial $\mathbf{MMS}(K, N)$

In 2006, Jaroslaw Wroblewski found the first nontrivial $\mathbf{MMS}(2, 6)$.

Examples of nontrivial $\mathbf{MMS}(K, N)$

In 2006, Jaroslaw Wroblewski found the first nontrivial $\mathbf{MMS}(2, 6)$.

17	36	55	124	62	114
58	40	129	50	111	20
108	135	34	44	38	49
87	98	92	102	1	28
116	25	86	7	96	78
22	74	12	81	100	119

Examples of nontrivial **MMS**(K, N)

In 2006, Jaroslaw Wroblewski found the first nontrivial **MMS**$(2, 6)$.

17	36	55	124	62	114
58	40	129	50	111	20
108	135	34	44	38	49
87	98	92	102	1	28
116	25	86	7	96	78
22	74	12	81	100	119

Open Problem

Does there exist a nontrivial **MMS**$(2, 5)$?

Examples of nontrivial $\mathbf{MMS}(K, N)$

In 2002, Walter Trump found the first nontrivial $\mathbf{MMS}(3, 12)$.

Examples of nontrivial **MMS**(K, N)

In 2002, Walter Trump found the first nontrivial **MMS**$(3, 12)$.

1	22	33	41	62	66	79	83	104	112	123	144
9	119	45	115	107	93	52	38	30	100	26	136
75	141	35	48	57	14	131	88	97	110	4	70
74	8	106	49	12	43	102	133	96	39	137	71
140	101	124	42	60	37	108	85	103	21	44	5
122	76	142	86	67	126	19	78	59	3	69	23
55	27	95	135	130	89	56	15	10	50	118	90
132	117	68	91	11	99	46	134	54	77	28	13
73	64	2	121	109	32	113	36	24	143	81	72
58	98	84	116	138	16	129	7	29	61	47	87
80	34	105	6	92	127	18	53	139	40	111	65
51	63	31	20	25	128	17	120	125	114	82	94

Not only is this **MMS**$(3, 12)$ nontrivial, it is normal!

Examples of nontrivial **MMS**(K, N)

In 2002, Walter Trump found the first nontrivial **MMS**$(3, 12)$.

1	22	33	41	62	66	79	83	104	112	123	144
9	119	45	115	107	93	52	38	30	100	26	136
75	141	35	48	57	14	131	88	97	110	4	70
74	8	106	49	12	43	102	133	96	39	137	71
140	101	124	42	60	37	108	85	103	21	44	5
122	76	142	86	67	126	19	78	59	3	69	23
55	27	95	135	130	89	56	15	10	50	118	90
132	117	68	91	11	99	46	134	54	77	28	13
73	64	2	121	109	32	113	36	24	143	81	72
58	98	84	116	138	16	129	7	29	61	47	87
80	34	105	6	92	127	18	53	139	40	111	65
51	63	31	20	25	128	17	120	125	114	82	94

Not only is this **MMS**$(3, 12)$ nontrivial, it is normal!

Open Problem

Does there exist a nontrivial **MMS**$(3, 11)$?

Formulating our problem

Given $K \geqslant 2$ let $N(K)$ denote the smallest natural number for which there exists non-trivial $\mathbf{MMS}(K, N(K))$.

Formulating our problem

Given $K \geqslant 2$ let $N(K)$ denote the smallest natural number for which there exists non-trivial $\mathbf{MMS}(K, N(K))$.

K	Upper bound on $N(K)$	Attributed to
2	6	J. Wroblewski
3	12	W. Trump
4	243	P. Fengchu
5	729	L. Wen
6	4096	P. Fengchu
$K \geqslant 2$	$(4K-2)^K$	Zhang, Chen, and Lei

Our results

Via the Hardy-Littlewood circle method we establish

Theorem (F. 2024+)

$N(K) \leqslant 2K(K+1)+1$ for $K \geqslant 2$.

Our results

Via the Hardy-Littlewood circle method we establish

Theorem (F. 2024+)

$N(K) \leqslant 2K(K+1)+1$ for $K \geqslant 2$.

- This beats previously known results as soon as $K \geqslant 4$ and shows that $N(K)$ grows at most quadratically in K rather than potentially exponential in K.

Our results

- One may prove an analogous statement for prime valued **MMS**(K, N) by reapplying the entirety of the circle method where we detect prime solutions instead of integer solutions.

Our results

- One may prove an analogous statement for prime valued $\mathbf{MMS}(K, N)$ by reapplying the entirety of the circle method where we detect prime solutions instead of integer solutions.
- This, however, is not necessary as via an argument due to Granville in his expository article *Prime Number Patterns* one may apply the Green-Tao theorem and deduce the following.

Our results

- One may prove an analogous statement for prime valued $\mathbf{MMS}(K, N)$ by reapplying the entirety of the circle method where we detect prime solutions instead of integer solutions.
- This, however, is not necessary as via an argument due to Granville in his expository article *Prime Number Patterns* one may apply the Green-Tao theorem and deduce the following.

Corollary (F. 2024+)

Given $K \geqslant 2$ there exists infinitely many nontrivial prime valued $\mathbf{MMS}(K, N)$ for every $N > 2K(K+1)$.

Brief overview of our methods

Let $C = (c_{i,j})_{\substack{1\leqslant i\leqslant r\\ 1\leqslant j\leqslant s}} \in \mathbb{Z}^{r\times s}$ be given, and consider the diagonal system

$$\sum_{1\leqslant j\leqslant s} c_{i,j}x_j^k = 0 \quad (1\leqslant i\leqslant r, \quad 1\leqslant k\leqslant K). \tag{1.1}$$

We define $R_K(P;C)$ to be the number of solutions $\mathbf{x}\in\mathbb{Z}^s$ to (1.1) where $\max_j |x_j| \leqslant P$.

Brief overview of our methods

Let $C = (c_{i,j})_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}} \in \mathbb{Z}^{r \times s}$ be given, and consider the diagonal system

$$\sum_{1 \leqslant j \leqslant s} c_{i,j} x_j^k = 0 \quad (1 \leqslant i \leqslant r, \quad 1 \leqslant k \leqslant K). \tag{1.1}$$

We define $R_K(P; C)$ to be the number of solutions $\mathbf{x} \in \mathbb{Z}^s$ to (1.1) where $\max_j |x_j| \leqslant P$.
There exists of a matrix $C_N^{\text{magic}} \in \{-1, 0, 1\}^{2N \times N^2}$ for which $R_K(P; C_N^{\text{magic}})$ counts the number of $\mathbf{MMS}(K, N)$ with entries satisfying

$$\max_{1 \leqslant i,j \leqslant N} |z_{i,j}| \leqslant P.$$

Brief overview of our methods

The number of trivial $\mathbf{MMS}(K, N)$ counted by $R_K(P; C_N^{\text{magic}})$ is at most $O(P^N)$, thus if one wishes to establish the existence of non-trivial $\mathbf{MMS}(K, N)$ it is enough to show that

$$\frac{R_K(P; C_N^{\text{magic}})}{P^N} \to \infty \quad \text{as} \quad N \to \infty. \tag{1.2}$$

Brief overview of our methods

The number of trivial **MMS**(K, N) counted by $R_K(P; C_N^{\text{magic}})$ is at most $O(P^N)$, thus if one wishes to establish the existence of non-trivial **MMS**(K, N) it is enough to show that

$$\frac{R_K(P; C_N^{\text{magic}})}{P^N} \to \infty \quad \text{as} \quad N \to \infty. \tag{1.2}$$

Theorem (F. 2024+)

For $K \geqslant 2$ and $N > 2K(K+1)$ there exists a constant $c > 0$ for which one has the asymptotic formula

$$R_K(P; C_N^{\text{magic}}) \sim cP^{N(N-K(K+1))}.$$

Brief overview of our methods

Given C estimating $R_K(P; C)$ via the circle method has been done in various general cases by several other mathematicians such as Brandes, Parsell, and Wooley.

Brief overview of our methods

Given C estimating $R_K(P; C)$ via the circle method has been done in various general cases by several other mathematicians such as Brandes, Parsell, and Wooley.

However, in their application of the circle method they require the $r \times s$ matrix of coefficients C to be *highly non-singular*, i.e., for all $J \subset \{1, \ldots, s\}$ with $|J| = r$ one should have

$$\det \left(c_{i,j}\right)_{\substack{1 \leqslant i \leqslant r \\ j \in J}} \neq 0.$$

Brief overview of our methods

Given C estimating $R_K(P; C)$ via the circle method has been done in various general cases by several other mathematicians such as Brandes, Parsell, and Wooley.
However, in their application of the circle method they require the $r \times s$ matrix of coefficients C to be *highly non-singular*, i.e., for all $J \subset \{1, \ldots, s\}$ with $|J| = r$ one should have

$$\det (c_{i,j})_{\substack{1 \leqslant i \leqslant r \\ j \in J}} \neq 0.$$

Our matrix of interest C_N^{magic} does not satisfy this property. Thus we developed a version of the circle method for this problem that works if C satisfies some weaker property.

Brief overview of our methods

For a given $r \times s$ matrix $C = [\mathbf{c}_1, \ldots, \mathbf{c}_s]$ and any set $J \subset \{1, \ldots, s\}$, we denote by C_J the submatrix of C consisting of the columns $\mathbf{c}_j$ where $j \in J$.

Brief overview of our methods

For a given $r \times s$ matrix $C = [\mathbf{c}_1, \ldots, \mathbf{c}_s]$ and any set $J \subset \{1, \ldots, s\}$, we denote by C_J the submatrix of C consisting of the columns $\mathbf{c}_j$ where $j \in J$.
For any $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ we denote by $\text{rem}(a, b)$ the remainder of a modulo b considered as an integer between 0 and $b - 1$.

For a given $r \times s$ matrix $C = [\mathbf{c}_1, \ldots, \mathbf{c}_s]$ and any set $J \subset \{1, \ldots, s\}$, we denote by C_J the submatrix of C consisting of the columns $\mathbf{c}_j$ where $j \in J$.
For any $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ we denote by $\operatorname{rem}(a, b)$ the remainder of a modulo b considered as an integer between 0 and $b - 1$.

Definition

We say that a matrix C dominates a function $f : \mathbb{N} \to \mathbb{R}^+$ whenever the inequality

$$\operatorname{rank}(C_J) \geqslant \min\{f(|J|), r\},$$

holds for all $J \subset \{1, \ldots, s\}$.

Brief overview of our methods

Quantitative Hasse Principle (F. 2024+)

Let $K \geqslant 2$ and suppose that $C \in \mathbb{Z}^{r\times s}$ satisfies $s > rK(K+1)$. Then, if C dominates the function

$$F(x) = \max\left\{ \frac{x - \mathrm{rem}(s,r)}{\left\lfloor \frac{s}{r} \right\rfloor}, \frac{x - \mathrm{rem}(s-1,r)}{\left\lfloor \frac{s-1}{r} \right\rfloor} \right\},$$

one has that

$$R_K(P;C) = P^{s - \frac{rK(K+1)}{2}} \left(\sigma_K(C) + o(1)\right),$$

where $\sigma_K(C) \geqslant 0$ is a real number depending only on K and C. Additionally $\sigma_K(C) > 0$ if there exists non-singular real and p-adic solutions to the system (1.1).

Brief overview of our methods

Thus there are two things left to establish.

Brief overview of our methods

Thus there are two things left to establish.

- The matrix C_N^{magic} dominates the the function $F(x)$ defined previously.

Brief overview of our methods

Thus there are two things left to establish.

- The matrix C_N^{magic} dominates the the function $F(x)$ defined previously.
- This is done via a combinatorial argument and understanding the underlying linear system associated to the matrix C_N^{magic}.

Brief overview of our methods

Thus there are two things left to establish.

- The matrix C_N^{magic} dominates the the function $F(x)$ defined previously.
- This is done via a combinatorial argument and understanding the underlying linear system associated to the matrix C_N^{magic}.
- Establish the existence of nonsingular real and p-adic $\mathbf{MMS}(K, N)$.

Brief overview of our methods

Thus there are two things left to establish.

- The matrix C_N^{magic} dominates the the function $F(x)$ defined previously.
- This is done via a combinatorial argument and understanding the underlying linear system associated to the matrix C_N^{magic}.
- Establish the existence of nonsingular real and p-adic $\mathbf{MMS}(K, N)$.
- This is done by looking at a particular integer valued $\mathbf{MMS}(K, N)$ and showing that the Jacobian of the associated linear system at that that point is full rank.

Breaking news!

[Submitted on 13 Jun 2024]

On the existence of magic squares of powers

Nick Rome, Shuntaro Yamagishi (with an appendix by Diyuan Wu)

Breaking news!

[Submitted on 13 Jun 2024]

On the existence of magic squares of powers

Nick Rome, Shuntaro Yamagishi (with an appendix by Diyuan Wu)

Theorem 1.3. *Let $d \geqslant 3$. There is a positive integer $n_0(d)$ such that for every integer $n \geqslant n_0(d)$, there exists an $n \times n$ magic square of d^{th} powers.*

$$n_0(d) = \begin{cases} 14\min\{2^d, d(d+1)\} + 79 & \text{if } 3 \leqslant d \leqslant 19, \\ 14\lceil d(\log d + 4.20032)\rceil + 79 & \text{if } d \geqslant 20. \end{cases}$$

Breaking news!

[Submitted on 13 Jun 2024]

On the existence of magic squares of powers

Nick Rome, Shuntaro Yamagishi (with an appendix by Diyuan Wu)

Theorem 1.3. *Let $d \geqslant 3$. There is a positive integer $n_0(d)$ such that for every integer $n \geqslant n_0(d)$, there exists an $n \times n$ magic square of d^{th} powers.*

$$n_0(d) = \begin{cases} 14\min\{2^d, d(d+1)\} + 79 & \text{if } 3 \leqslant d \leqslant 19, \\ 14\lceil d(\log d + 4.20032)\rceil + 79 & \text{if } d \geqslant 20. \end{cases}$$

What would our methods say if applied to this single degree case?

Single degree case

Sugar

1992 paper of Brüdern and Cook, *On simultaneous diagonal equations and inequalities*.

Single degree case

Sugar

1992 paper of Brüdern and Cook, *On simultaneous diagonal equations and inequalities*.

Spice

Modern smooth exponential sum bounds.

Single degree case

Sugar

1992 paper of Brüdern and Cook, *On simultaneous diagonal equations and inequalities*.

Spice

Modern smooth exponential sum bounds.

Everything not nice

Our result on C_N^{magic} dominating the the function $F(x)$

Single degree case

Sugar

1992 paper of Brüdern and Cook, *On simultaneous diagonal equations and inequalities*.

Spice

Modern smooth exponential sum bounds.

Everything not nice

Our result on C_N^{magic} dominating the the function $F(x)$

Confident can show

For $d \geqslant 3$ there exists a nontrivial $N \times N$ square of dth powers for all $N \geqslant d(\log(d) + 4.20032) + 1$.

Thank you for listening!