A Quantitative Hasse principle for Weighted Quartic Forms

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- Whenever the notation $|\mathbf{x}|$ is used for a vector we mean the maximum absolute value the elements of $\mathbf{x}$.
- As is conventional in analytic number theory, we write $e(z)$ for $e^{2\pi iz}$.

Hasse principle

Definition

A polynomial $F \in \mathbb{Z}[x_1, \ldots, x_s]$ is said to be an *integral homogeneous form* of degree d if for all $\lambda \in \mathbb{C}$ it satisfies the equation

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We will say the Hasse principle holds for

$$F(\mathbf{x}) = 0, \tag{1.1}$$

if the existence of solutions to (1.1) in $(\mathbb{R}\backslash 0)^s$ and $(\mathbb{Q}_p)\backslash 0)^s$ for every prime p implies the existence of a solution to (1.1) where $\mathbf{x} \in (\mathbb{Z}\backslash 0)^s$.

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Theorem (Birch,1961)

If F is a non-singular form of degree d in s variables then it satisfies the Hasse principle whenever $s > (d-1)2^d$.

Weighted Homogeneous Forms

Definition

We define a polynomial $F \in \mathbb{Z}[x_1, \ldots, x_s]$ to be an *integral weighted form* of degree d if there exists a vector $\mathbf{w} \in \mathbb{N}^s$, satisfying $(w_1, \ldots, w_s) = 1$, for which the equation

$$F(\lambda^{w_1} x_1, \ldots, \lambda^{w_s} x_s) = \lambda^d F(x_1, \ldots, x_s),$$

holds for all complex λ. With this notation we say that the variable x_i has weight w_i.

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Question

What can be said about weighted integral forms?

Trivial Considerations

Let $F(\mathbf{x})$ be a weighted form of degree d with weights $\mathbf{w}$, then trivially

$$G(\mathbf{x}) = F(x_1^{w_1}, \ldots, x_s^{w_s})$$

is a homogeneous form of degree d. One may then apply Birch's results and conclude that G (and hence F) satisfies the Hasse principle whenever $s > (d-1)2^d$.

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Goal

Use the weights $\mathbf{w}$ to establish a non-trivial statement about weighted forms.

Work Towards This Goal

Let $d \geqslant 2$ be given, then if we let $H^{(n)}$ denote a form of degree n one may investigate the zeros of the weighted form of degree $2d$, $F_{2d} \in Z[x_1, \ldots, x_{s_1}, y_1, \ldots, y_{s_2}]$ defined by

$$F_{2d}(\mathbf{x}; \mathbf{y}) = H^{(2)}(\mathbf{x}) + \sum_{1 \leqslant i \leqslant s_1} x_i H_i^{(d)}(\mathbf{y}) + H^{(2d)}(\mathbf{y}).$$

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Notice

F_{2d} is a general weighted form of degree $2d$ in which the variable weights are either 1 or d.

Key Idea

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Since quadratic forms are diagonalizable and one may complete the square to separate variables of different weights. Hence, there exists a non-singular change of variables $\mathbf{x} \mapsto \mathbf{u}$ and $\mathbf{y} \mapsto \mathbf{v}$ and a homogeneous polynomial $\tilde{H}^{(2d)}$ of degree $2d$ for which F_{2d} is an integer multiple of

$$\sum_{1 \leqslant i \leqslant s_1} a_i u_i^2 + \tilde{H}^{(2d)}(\mathbf{v}).$$

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Theorem Part 1 (F, 2024)

Let $d = 2$ and $F_{2d} \in Z[x_1, \ldots, x_{s_1}, y_1, \ldots, y_{s_2}]$ be as previously defined, and suppose the polynomials one gets after our change of variables $\tilde{H}^{(2d)}$ are non-degenerate binary forms. Then

$$F_{2d}(\mathbf{x}; \mathbf{y}) = 0$$

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or

$$s_1/2 + s_2/4 > 2 \quad \text{when} \quad s_1 \geqslant 2.$$

Our Results

Let
$R(X;\mathfrak{B}_1,\mathfrak{B}_2) = \#\{(\mathbf{x},\mathbf{y}) \in X^{1/2}\mathfrak{B}_1 \times X^{1/4}\mathfrak{B}_2 : F_{2d}(\mathbf{x};\mathbf{y}) = 0\}$

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Theorem Part 2 (F, 2024)

Let $F_{2d} \in Z[x_1,\ldots,x_{s_1},y_1,\ldots,y_{s_2}]$ satisfy the previous conditions. Then if $F_{2d}(\mathbf{x}) = 0$ contains non-trivial non-singular local solutions there exists boxes $\mathfrak{B}_1 \in [-1,1]^{s_1}$ and $\mathfrak{B}_2 \in [-1,1]^{s_2}$ for which one has the asymptotic formula

$$R(X;\mathfrak{B}_1,\mathfrak{B}_2) = X^{s_1/2+s_2/4-1}(\sigma + o(1)),$$

where $\sigma > 0$ is the expected product of local densities.

Proof Method

- We estimate $R(X;\mathfrak{B}_1,\mathfrak{B}_2)$ via the Hardy-Littlewood circle method and write

$$R(X;\mathfrak{B}_1,\mathfrak{B}_2)=\int_0^1 \sum_{\substack{x\in X^{1/2}\mathfrak{B}_1\\ y\in X^{1/4}\mathfrak{B}_2}} e(\alpha F_{2d}(\mathbf{x};\mathbf{y}))\,\mathrm{d}\alpha$$

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- As is standard the minor arcs are the difficult part and require the development of two key lemmas.

Summary of First Key Lemma

Lemma #1

Under certain conditions one may bound the two dimensional exponential sum

$$\sum_{\substack{1\leqslant x\leqslant P\\ 1\leqslant y\leqslant Q}} e(\alpha(x^2 + xH^{(d)}(\mathbf{y}) + H^{(2d)}(\mathbf{y}))),$$

via a bound on

$$(PQ)^{\varepsilon}\left(\sum_{|x|\ll P} e(\alpha x^2)\right)\left(\sum_{|y|\ll Q} e(\alpha \tilde{H}^{(2d)}(\mathbf{y}))\right).$$

This was shown via the previously mentioned completing the squares technique and an application of standard bounds on linear exponential sums.

Second Key Lemma

Let $G^{(1)} \in \mathbb{Q}$, $G^{(2)} \in \mathbb{R}$. Also, let $H^{(1)} \in \mathbb{Q}[y_1, y_2]$ be a non-degenerate homogeneous quartic, and $H^{(2)} \in \mathbb{R}[y_1, y_2]$ be a non-degenerate homogeneous quadratic. Then for positive numbers P, Q, we define the exponential sums

$$g(\alpha) = \sum_{|x| \leqslant P} e(\alpha G^{(1)} x^2 + G^{(2)} x),$$

$$h(\alpha) = \sum_{|\mathbf{y}| \leqslant Q} e\left(\alpha H^{(1)}(\mathbf{y}) + H^{(2)}(\mathbf{y})\right).$$

Then for large P, Q we have the following mean value bound

$$\|g(\alpha)^2 h(\alpha)\|_2^2 \ll PQ^{2+\varepsilon} + Q^4 P^{\varepsilon},$$

where the implicit constant is dependent only on $G^{(1)}$ and the coefficients of $H^{(1)}$.

Proof Sketch of Key Lemma #2

Simplification: Assume

$$g(\alpha) = \sum_{|x| \leqslant P} e(\alpha x^2), \text{ and } \quad h(\alpha) = \sum_{|\mathbf{y}| \leqslant Q} e(\alpha(y_1^4 + y_2^4)),$$

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then by orthogonality one has that $\|g(\alpha)^2 h(\alpha)\|_2^2$ counts the number of integers $x_1, x_2, y_1, y_2, y_3, y_4$ satisfying $|x_1|, |x_2| \leqslant P$ and $|y_1|, \ldots, |y_4| \leqslant Q$ for which

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- Case 2: If $|x_1| \neq |x_2|$ then there are trivially at most $(2Q)^4$ choices for $y_1, \ldots, y_4$. Since

$$x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2) \neq 0,$$

  we may use a divisor estimate to deduce the number of choices for x_1, x_2 is bounded by P^{ε}.

Future Pursuits

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Future Pursuits

- Question: Can we prove a general analogue of Birch's theorem for weighted homogeneous forms where we make use of the weights $\mathbf{w}$?
- Partial Answer: This problem appears to be beyond the scope of our current techniques as we require some way to separate the variables of large weight.
- Reasonable Question: What other classes of weighted homogeneous forms can we investigate and use of the weights $\mathbf{w}$ to save on the number of variables required by Birch's Theorem.

Thank you for listening!