

# THE HASSE PRINCIPLE FOR RANDOM HOMOGENEOUS POLYNOMIALS IN THIN SETS

DANIEL FLORES AND KISEOK YEON

ABSTRACT. Let  $d$  and  $n$  be natural numbers. Let  $\nu_{d,n} : \mathbb{R}^n \rightarrow \mathbb{R}^N$  denote the Veronese embedding with  $N = N_{n,d} := \binom{n+d-1}{d}$ , defined by listing all the monomials of degree  $d$  in  $n$  variables using the lexicographical ordering. Let  $\langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle \in \mathbb{Z}[\mathbf{x}]$  be a homogeneous polynomial in  $n$  variables of degree  $d$  with integer coefficients  $\mathbf{a}$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product. For a non-singular form  $P \in \mathbb{Z}[\mathbf{x}]$  of degree  $k$  ( $\leq d$ ) in  $N$  variables, consider a set of integer vectors  $\mathbf{a} \in \mathbb{Z}^N$ , defined by

$$\mathfrak{A}(A; P) = \{\mathbf{a} \in \mathbb{Z}^N : P(\mathbf{a}) = 0, \|\mathbf{a}\|_\infty \leq A\}.$$

By handling a new lattice problem via the geometry of numbers, we confirm that whenever  $n > 24d$  and  $d \geq 17$ , the proportion of integer coefficients  $\mathbf{a} \in \mathfrak{A}(A; P)$ , whose associated equation  $f_{\mathbf{a}}(\mathbf{x}) = 0$  satisfies the Hasse principle, converges to 1 as  $A \rightarrow \infty$ . This improves on the recent work of the second author.

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## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

This paper is concerned with the solubility for random diophantine equations. By [4] and [10], we see that the positive portion of homogeneous polynomials in a sufficiently large number of variables in terms of degree is soluble in  $\mathbb{Q}$ . To explain more precisely, we let  $\nu_{d,n} : \mathbb{R}^n \rightarrow \mathbb{R}^N$  denote the Veronese embedding with  $N = N_{d,n} := \binom{n+d-1}{d}$ , defined by listing all the monomials of degree  $d$  in  $n$  variables using the lexicographical ordering. Let  $\langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle \in \mathbb{Z}[\mathbf{x}]$  be a homogeneous polynomial in  $n$  variables of degree  $d$

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with integer coefficients  $\mathbf{a}$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product. Here and throughout this paper, we write  $f_{\mathbf{a}}(\mathbf{x}) = \langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle$ . Define

$$\mathfrak{A}(A) := \{\mathbf{a} \in \mathbb{Z}^N : \|\mathbf{a}\|_{\infty} \leq A\}.$$

Browning, Sawin and Le Boudec [4] proved that the proportion of  $\mathbf{a} \in \mathbb{Z}^N$  in  $\mathfrak{A}(A)$ , whose associated equation  $f_{\mathbf{a}}(\mathbf{x}) = 0$  satisfies the Hasse principle, converges to 1 as  $A \rightarrow \infty$ , provided that  $n \geq d + 1$  and  $d \geq 4$ . Furthermore, Poonen and Voloch [10] proved that the portion of  $\mathbf{a} \in \mathbb{Z}^N$  in  $\mathfrak{A}(A)$ , whose associated equation  $f_{\mathbf{a}}(\mathbf{x}) = 0$  is everywhere locally soluble, converges to a positive constant  $c$  as  $A \rightarrow \infty$ , provided that  $n \geq 2$  and  $d \geq 2$ . Hence, combining these two results, we conclude that the positive portion of homogeneous polynomial equations in  $n$  variables of degree  $d$  is soluble in  $\mathbb{Q}$ , provided that  $n \geq d + 1$  and  $d \geq 4$ .

The recent works ([9], [13]) of Lee, Lee and the second author showed that such a conclusion on global solubility remains true even when the coefficients  $\mathbf{a} \in \mathbb{Z}^N$  are constrained by a polynomial condition under a modest condition on the number of variables. More precisely, let  $P$  be a non-singular form in  $n$  variables of degree  $k \geq 2$ . Define

$$\mathfrak{A}(A; P) := \{\mathbf{a} \in \mathbb{Z}^N : P(\mathbf{a}) = 0, \|\mathbf{a}\|_{\infty} \leq A\}.$$

The second author [13] proved that the proportion of  $\mathbf{a} \in \mathbb{Z}^N$  in  $\mathfrak{A}(A; P)$ , whose associated equation  $f_{\mathbf{a}}(\mathbf{x}) = 0$  satisfies the Hasse principle, converges to 1 as  $A \rightarrow \infty$ , provided that  $n \geq 32d + 17$ ,  $d \geq 14$  and  $k \leq d$ . Furthermore, Lee, Lee and the second author [9] showed that the portion of  $\mathbf{a} \in \mathbb{Z}^N$  in  $\mathfrak{A}(A; P)$ , whose associated equation  $f_{\mathbf{a}}(\mathbf{x}) = 0$  is everywhere locally soluble, converges to a constant  $c_P$  as  $A \rightarrow \infty$ , provided that  $n \geq 2$ ,  $d \geq 2$  and  $k \leq C(n, d)$  for some constant  $C(n, d)$ . This constant  $c_P$  is positive if there exists  $\mathbf{a} \in \mathfrak{A}(A; P)$  such that  $f_{\mathbf{a}}(\mathbf{x}) = 0$  has an integer solution  $\mathbf{x} \in \mathbb{Z}^n$  with  $\nabla f_{\mathbf{a}}(\mathbf{x}) \neq \mathbf{0}$ .

The main purpose of this paper is to relax on the bound for the number of variables  $n \geq 32d + 17$  for such a conclusion. The crucial ingredient for this improvement is to handle a new lattice counting problem (see Lemma 3.1 below) which naturally arises in the argument. To do so, we use tools from the geometry of numbers. Our hope is that this argument described in this paper may be helpful to make further improvement.

In order to describe our main theorems, we temporarily pause here and provide some definitions. Recall that  $f_{\mathbf{a}}(\mathbf{x})$  is a homogeneous polynomial in  $n$  variables of degree  $d$ . Furthermore, for  $\mathbf{a} \in \mathbb{Z}^N$  and  $X > 0$ , we define

$$\mathcal{I}_{\mathbf{a}}(X) := \{\mathbf{x} \in [1, X]^n \cap \mathbb{Z}^n : f_{\mathbf{a}}(\mathbf{x}) = 0\}.$$

We note here that our argument proceeds for fixed  $X > 0$ , and thus for simplicity, we write

$$(1.1) \quad w = \log X$$

and

$$(1.2) \quad W = \prod_{p \leq w} p^{\lfloor \log w / \log p \rfloor}.$$

Observe here that an application of the prime number theorem reveals that  $\log W \leq 2w$ , which implies

$$(1.3) \quad W \leq X^2.$$

For  $L > 0$  and  $\mathbf{a} \in \mathbb{Z}^N$ , we define

$$(1.4) \quad \sigma(\mathbf{a}; L) = L^{-(n-1)} \#\{\mathbf{g} \in [1, L]^n : f_{\mathbf{a}}(\mathbf{g}) \equiv 0 \pmod{L}\}.$$

We notice that by the Chinese remainder theorem one has

$$(1.5) \quad \sigma(\mathbf{a}; L) = \prod_{p^r \parallel L} \sigma(\mathbf{a}; p^r).$$

Then, on recalling the definition (1.2) of  $W$ , we write

$$(1.6) \quad \mathfrak{S}_{\mathbf{a}}^* = \sigma(\mathbf{a}; W) = \prod_{p^r \parallel W} \sigma(\mathbf{a}; p^r).$$

Recall the definition (1.1) of  $w$ . Put  $\zeta = w^{-4-1/(8d)}$ , and we introduce an auxiliary function

$$\mathfrak{w}_{\zeta}(\beta) = \zeta \cdot \left( \frac{\sin(\pi\zeta\beta)}{\pi\zeta\beta} \right)^2.$$

Note here that we chose  $\zeta$  differently from [13] in order to optimize the result. This function has the Fourier transform

$$\widehat{\mathfrak{w}}_{\zeta}(\xi) = \int_{-\infty}^{\infty} \mathfrak{w}_{\zeta}(\beta) e(-\beta\xi) d\beta = \max\{0, 1 - |\xi|/\zeta\}.$$

For  $\mathbf{a} \in \mathbb{Z}^N$  and  $A, X > 0$ , we define

$$(1.7) \quad \mathfrak{J}_{\mathbf{a}}^* := \mathfrak{J}_{\mathbf{a}}^*(A, X) = A^{-1} X^{n-d} \int_{[0,1]^n} \zeta^{-1} \widehat{\mathfrak{w}}_{\zeta}(A^{-1} f_{\mathbf{a}}(\boldsymbol{\gamma})) d\boldsymbol{\gamma}.$$

**Definition 1.1.** *Let  $n$  and  $d$  be natural numbers with  $d \geq 2$ . Consider the monomials of degree  $d$  in  $n$  variables  $x_1, \dots, x_n$ . In particular, the number of these monomials is  $N = \binom{n+d-1}{d}$ . Then, define  $v_d(\mathbf{x}) \in \mathbb{R}^n$  and  $w_d(\mathbf{x}) \in \mathbb{R}^{N-n}$  to be vectors associated with those monomials such that  $(v_d(\mathbf{x}))_i$  is  $x_i^d$  with  $i = 1, \dots, n$  and  $(w_d(\mathbf{x}))_j$ 's are remaining monomials in lexicographical order with  $j = 1, \dots, N - n$ , respectively.*

For example, we find that

$$v_3(x_1, x_2) = (x_1^3, x_2^3) \text{ and } w_3(x_1, x_2) = (x_1^2 x_2, x_1 x_2^2).$$

Recall the definition of  $f_{\mathbf{a}}(\mathbf{x})$ , that is  $f_{\mathbf{a}}(\mathbf{x}) = \langle \mathbf{a}, \nu_{d,n}(\mathbf{x}) \rangle$ . Let us define a permutation  $[\cdot] : \mathbb{Z}^N \rightarrow \mathbb{Z}^N$  in the following way: for a given  $(\mathbf{b}, \mathbf{c}) \in \mathbb{Z}^N$

with  $\mathbf{b} \in \mathbb{Z}^n$  and  $\mathbf{c} \in \mathbb{Z}^{N-n}$ , the permutation  $[\cdot]$  is mapping  $(\mathbf{b}, \mathbf{c}) \in \mathbb{Z}^N$  to  $\mathbf{a} \in \mathbb{Z}^N$  such that

$$(1.8) \quad f_{\mathbf{a}}(\mathbf{x}) = \langle \mathbf{b}, v_d(\mathbf{x}) \rangle + \langle \mathbf{c}, w_d(\mathbf{x}) \rangle.$$

**Theorem 1.2.** *Let  $A$  and  $X$  be positive numbers and let  $n$  and  $d$  be natural numbers with  $d \geq 4$ . Let  $n = 8s + r$  with  $s \in \mathbb{N}$  and  $1 \leq r \leq 8$ . Suppose that  $s \geq 3d$  and  $X^{2d} \leq A \leq X^{s-d}$ . Suppose that  $P \in \mathbb{Z}[\mathbf{x}]$  is a non-singular form of degree  $k$  in  $N_{d,n}$  variables. Then, whenever  $N_{d,n} \geq 200k(k-1)2^{k-1}$ , there is a positive number  $\delta < 1$  such that*

$$\sum_{\substack{\|\mathbf{a}\|_{\infty} \leq A \\ P(\mathbf{a})=0}} |\mathcal{I}_{\mathbf{a}}(X) - \mathfrak{S}_{\mathbf{a}}^* \mathfrak{J}_{\mathbf{a}}^*|^2 \ll A^{N-4} X^{2n-2d} (\log A)^{-\delta}.$$

Recall the definition of  $N := N_{d,n}$ . In advance of the statement of the following theorem, define a set  $\mathcal{A}_{d,n}^{\text{loc}}(A; P)$  of integer vectors  $\mathbf{a} \in \mathbb{Z}^N$  in  $\mathfrak{A}(A; P)$  having the property that the associated equation  $f_{\mathbf{a}}(\mathbf{x}) = 0$  is everywhere locally soluble.

**Theorem 1.3.** *Let  $A$  and  $X$  be positive numbers with  $X^3 \leq A$ . Suppose that  $n$  and  $d$  are natural numbers with  $n > d + 1$  and  $d \geq 2$ . Suppose that  $P \in \mathbb{Z}[\mathbf{x}]$  is a non-singular form of degree  $k$  in  $N_{d,n}$  variables. Then, whenever  $N_{d,n} \geq 1000n^2 8^k$ , one has*

$$\#\left\{ \mathbf{a} \in \mathcal{A}_{d,n}^{\text{loc}}(A; P) : \mathfrak{S}_{\mathbf{a}}^* \mathfrak{J}_{\mathbf{a}}^* \leq X^{n-d} A^{-1} (\log A)^{-\eta} \right\} \ll A^{N-2} (\log A)^{-\eta/(40n)},$$

for any  $\eta > 0$ .

*Proof.* Only difference between Theorem 1.3 and [13, Theorem 1.2] is the choice of  $\zeta$ . One readily sees that the choice of  $\zeta = w^{-4-1/(8d)}$  does not harm the argument in [13, Proposition 5.12], and thus [13, Proposition 5.12] still holds with  $\zeta = w^{-4-1/(8d)}$ . Therefore, we see by [13, section 6] that [13, Theorem 1.2] holds with  $\zeta = w^{-4-1/(8d)}$ .  $\square$

**Theorem 1.4.** *Let  $A$  and  $X$  be positive numbers. Suppose that  $A, X, n$  and  $d$  satisfy the hypotheses in Theorem 1.2 and 1.3. Suppose that  $P \in \mathbb{Z}[\mathbf{x}]$  is a non-singular form of degree  $k$  in  $N_{d,n}$  variables. Then, the proportion of integer vectors  $\mathbf{a} \in \mathcal{A}_{d,n}^{\text{loc}}(A; P)$  in  $\mathfrak{A}(A; P)$ , having the property that*

$$\mathcal{I}_{\mathbf{a}}(X) < A^{-1} X^{n-d} (\log A)^{-1/5},$$

converges to 0 as  $A \rightarrow \infty$ .

*Proof.* See the proof of [13, Theorem 1.3].  $\square$

**Corollary 1.5.** *The conclusions of Theorem 1.2, 1.3 and 1.4 hold for  $d \geq 17$ ,  $k \leq d$  and  $n > 24d$  in place of the hypotheses on  $n, d$  and  $k$ .*

*Proof.* It suffices to show that the conditions  $d \geq 17$ ,  $k \leq d$  and  $n > 24d$  imply the hypotheses on  $n, d$  and  $k$  in Theorem 1.2, 1.3 and 1.4. For  $d \geq 17$ , a modicum of computation reveals that we have

$$1000 \cdot 8^d \leq \frac{25^{d-2}}{d^2}.$$

Then, we see that whenever  $d \geq 17$  and  $n > 24d$ , we obtain

$$1000 \cdot 8^d \leq \frac{1}{d^2} \cdot \left( \frac{n+d-1}{d} \right)^{d-2}.$$

Hence, it follows that whenever  $k \leq d$  one has

$$1000 \cdot n^2 \cdot 8^k \leq 1000 \cdot 8^d \cdot (n+d-1)^2 \leq \left( \frac{n+d-1}{d} \right)^d \leq \binom{n+d-1}{d} = N_{d,n}.$$

Furthermore, it implies that  $N_{d,n} \geq 200k(k-1)2^{k-1}$ . Plus, if one writes  $n = 8s + r$  with  $1 \leq r \leq 8$ , one sees that  $s \geq 3d$  since  $n > 24d$ .  $\square$

Therefore, Theorem 1.4 implies that the proportion of integer vectors  $\mathbf{a} \in \mathbb{Z}^N$  in  $\mathfrak{A}(A; P)$ , whose associated equation  $f_{\mathbf{a}}(\mathbf{x}) = 0$  satisfies the Hasse principle, converges to 1. Meanwhile, by [9], one finds that the proportion of integer vectors  $\mathbf{a} \in \mathbb{Z}^N$  in  $\mathfrak{A}(A; P)$ , whose associated equation  $f_{\mathbf{a}}(\mathbf{x}) = 0$  is everywhere locally soluble, converges to a constant  $c_P$  as  $A \rightarrow \infty$ . Moreover, for each place  $v$  of  $\mathbb{Q}$ , if there exists a non-zero  $\mathbf{b}_v \in \mathbb{Q}_v^N$  such that  $P(\mathbf{b}_v) = 0$  and the equation  $f_{\mathbf{b}_v}(\mathbf{x}) = 0$  has a solution  $\mathbf{x} \in \mathbb{Q}_v^n$  satisfying  $\nabla f_{\mathbf{b}_v}(\mathbf{x}) \neq \mathbf{0}$  (see also [3] for  $k = 2$ ). Combining this with Theorem 1.4, we conclude that the proportion of integer vectors  $\mathbf{a} \in \mathbb{Z}^N$  in  $\mathfrak{A}(A; P)$ , whose associated equation  $f_{\mathbf{a}}(\mathbf{x}) = 0$  has a solution  $\mathbf{x} \in \mathbb{Q}^n$ , converges to  $c_P$ .

In this paper, we use bold symbol to denote vectors, and we write vectors by row vectors. For a given vector  $\mathbf{v} \in \mathbb{R}^N$ , we write  $i$ -th coordinate of  $\mathbf{v}$  by  $(\mathbf{v})_i$  or  $v_i$ . Given that  $n_1, n_2, \dots, n_s \in \mathbb{N}$  with  $n_1 + n_2 + \dots + n_s = n$ , we use notation  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s) \in \mathbb{R}^n$  with  $\mathbf{x}_1 \in \mathbb{R}^{n_1}$ ,  $\mathbf{x}_2 \in \mathbb{R}^{n_2}$ ,  $\dots$ ,  $\mathbf{x}_s \in \mathbb{R}^{n_s}$  where  $\mathbf{x}_1$  is a vector formed by the first  $n_1$  coordinates of  $\mathbf{x}$  and  $\mathbf{x}_2$  is a vector formed by the next  $n_2$  coordinates of  $\mathbf{x}$  and so on. We use notation  $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(l)}$  for

$$\left( \sum_{\mathbf{x}} \right)^s = \sum_{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(l)}}.$$

We write  $0 \leq \mathbf{x} \leq X$  to abbreviate the condition  $0 \leq x_1, \dots, x_s \leq X$ . Also, we preserve summation conditions until different conditions are specified.

## 2. AUXILIARY LEMMAS

In this section, we record some results from the geometry of numbers, and some results in [13]. Let  $\Lambda$  be a sublattice of  $\mathbb{Z}^n$  of rank  $r$ , and let  $\mathbf{b}_1, \dots, \mathbf{b}_r$  be its basis. Denote by  $d(\Lambda)$  the determinant of the lattice  $\Lambda$ . It follows by [Lemma IV.6A and 6D, 11] that

$$d(\Lambda)^2 = \sum_I (\det B_I)^2,$$

where  $I$  runs over  $r$ -element subsets of  $\{1, 2, \dots, n\}$ , and  $B_I$  denotes the  $r \times r$ -minor with rows indexed in  $I$  of the matrix  $B = (\mathbf{b}_1, \dots, \mathbf{b}_r)$  formed with columns  $\mathbf{b}_j$ . Define the orthogonal lattice  $\Lambda^\perp := \{\mathbf{x} \in \mathbb{Z}^n : \langle \mathbf{x}, \mathbf{b}_i \rangle = 0 \ (1 \leq i \leq r)\}$ . Define

$$G(\Lambda) := \gcd \det B_I.$$

Then, we find from [5, Lemma 2.1] that

$$(2.1) \quad d(\Lambda^\perp) = d(\Lambda)/G(\Lambda).$$

**Lemma 2.1.** *Let  $\Lambda$  be a sublattice of rank  $r$  in  $\mathbb{Z}^n$ . Let  $A \geq d(\Lambda)$ . Then, the box  $|\mathbf{a}| \leq A$  contains  $O(A^r/d(\Lambda))$  elements of  $\Lambda$ .*

*Proof.* See [Lemma 1 (v), 8]. □

**Lemma 2.2.** *Let  $U_s(A, X)$  be the number of integer solutions  $0 < |a_i| < A$  and  $0 \leq |x_i|, |z_i| \leq X$  satisfying*

$$\sum_{i=1}^s a_i (x_i^d - z_i^d) = 0.$$

*Then, we have*

$$U_s(A, X) \ll A^s X^s + A^{s-1} X^{2s-d+\epsilon}.$$

*Proof.* See [6, Theorem 2.5] □

To describe the next lemma, it is convenient to introduce some definitions. For  $\mathbf{h} \in \mathbb{Z}^N$ , we define

$$I_{\mathbf{h}} = \{\mathbf{a} \in \mathbb{Z}^N : \|\mathbf{a}\|_\infty \leq A, \|\mathbf{a} + \mathbf{h}\|_\infty \leq A\}.$$

Furthermore, we define

$$F_1(\alpha_1, \alpha_2, \mathbf{h}) = \sum_{\substack{1 \leq \mathbf{x}, \mathbf{y} \leq X \\ \mathbf{x}, \mathbf{y} \in \mathbb{Z}^n}} e(\alpha_1 \langle \mathbf{h}, \nu_{d,n}(\mathbf{x}) - \alpha_2 \langle \mathbf{h}, \nu_{d,n}(\mathbf{y}) \rangle)$$

and

$$F_2(\beta, \mathbf{h}) = \sum_{\mathbf{a} \in I_{\mathbf{h}}} e(\beta(P(\mathbf{a} + \mathbf{h}) - P(\mathbf{a}))).$$

Additionally, we define

$$G_1(\alpha_1, \alpha_2) = \sum_{\substack{\|\mathbf{h}\|_\infty \leq 2A \\ \mathbf{h} \in \mathbb{Z}^N}} |F_1(\alpha_1, \alpha_2, \mathbf{h})|^2 \text{ and } G_2(\beta) = \sum_{\substack{\|\mathbf{h}\|_\infty \leq 2A \\ \mathbf{h} \in \mathbb{Z}^N}} |F_2(\beta, \mathbf{h})|^2.$$

For any measurable set  $\mathfrak{B} \in [0, 1)$ ,  $\mathbf{a} \in \mathbb{Z}^N$  and  $X > 1$ , define

$$(2.2) \quad \mathcal{I}_{\mathbf{a}}(X, \mathfrak{B}) = \int_{\mathfrak{B}} \sum_{1 \leq x \leq X} e(\alpha f_{\mathbf{a}}(x)) d\alpha.$$

**Lemma 2.3.** *For any measurable set  $\mathfrak{B} \in [0, 1)$ , we have*

$$\sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{I}_{\mathbf{a}}(X, \mathfrak{B})|^2 \ll X^n \int_{\mathfrak{B}^2} G_1(\alpha_1, \alpha_2)^{1/4} d\alpha_1 d\alpha_2 \int_0^1 G_2(\beta)^{1/4} d\beta.$$

*Proof.* See [13, Lemma 3.5]. □

### 3. A LATTICE COUNTING PROBLEM

In this section, we provide a new lattice counting problem which is essential ingredient for the main theorem in this paper. Let  $N(A, X)$  denote the number of solutions of equations

$$(3.1) \quad \sum_{1 \leq i \leq s} a_i(x_i^d - z_i^d) = \sum_{1 \leq i \leq s} a_i(y_i^d - w_i^d) = 0$$

in integers  $a_i, x_i, y_i, z_i, w_i$  with  $|a_i| \leq A$  and  $|x_i|, |y_i|, |z_i|, |w_i| \leq X$ .

**Lemma 3.1.** *Let  $d \geq 4$  and  $s \geq d + 2$ . Suppose that  $A \geq X^{2d} \geq 1$ . Then, one has*

$$(3.2) \quad N(A, X) \ll A^s X^{2s} + A^{s-1} X^{3s-d+\varepsilon} + A^{s-2} X^{4s-2d+\varepsilon} E(X),$$

where

$$E(X) = 1 + X^{2d-s+3} + X^{4d-2s+4}.$$

*Proof.* We begin by observing that it suffices to count the number of solutions of equations:

$$(3.3) \quad \sum_{1 \leq i \leq s} a_i(x_i^d - z_i^d) = \sum_{1 \leq i \leq s} a_i(y_i^d - w_i^d) = 0$$

in integers  $a_i, x_i, y_i, z_i, w_i$  with  $0 < |a_i| \leq A$  and  $X/2 \leq |x_i|, |y_i|, |z_i|, |w_i| \leq X$ . Indeed, we temporarily define

$$W(\alpha_1, \alpha_2) := \sum_{|l| \leq A} \left| \sum_{|x| \leq X} e(\alpha_1 l x^d) \right|^2 \left| \sum_{|y| \leq X} e(\alpha_2 l y^d) \right|^2.$$

Then, it follows by orthogonality that

$$(3.4) \quad N(A, X) = \int_0^1 \int_0^1 |W(\alpha_1, \alpha_2)|^s d\alpha_1 d\alpha_2.$$

By the elementary inequality  $(a + b)^n \ll a^n + b^n$  with  $a, b \geq 0$  and  $n \in \mathbb{N}$ , we deduce from (3.4) that

$$(3.5) \quad N(A, X) \ll X^{4s} + A^s X^{2s} + \int_0^1 \int_0^1 |\widetilde{W}(\alpha_1, \alpha_2)|^s d\alpha_1 d\alpha_2,$$

where

$$\widetilde{W}(\alpha_1, \alpha_2) := \sum_{0 < |l| \leq A} \left| \sum_{1 \leq |x| \leq X} e(\alpha_1 l x^d) \right|^2 \left| \sum_{1 \leq |y| \leq X} e(\alpha_2 l y^d) \right|^2.$$

Let  $\widetilde{W}_{ij}(\alpha_1, \alpha_2)$  be the portion of the sum  $\widetilde{W}(\alpha_1, \alpha_2)$  where  $2^{-i}X < |x| \leq 2^{-i+1}X$  and  $2^{-j}X < |y| \leq 2^{-j+1}X$ . Then, by applying the Cauchy-Schwarz inequality, one has

$$\widetilde{W}(\alpha_1, \alpha_2) \ll X^\epsilon \sum_{i=1}^L \sum_{j=1}^L \widetilde{W}_{ij}(\alpha_1, \alpha_2)$$

with  $L = O(\log X)$ . Then, it following by applying the Hölder's inequality in (3.5) that

$$(3.6) \quad N(A, X) \ll X^{4s} + A^s X^{2s} + X^\epsilon \sum_{i=1}^L \sum_{j=1}^L \int_0^1 \int_0^1 |\widetilde{W}_{ij}(\alpha_1, \alpha_2)|^s d\alpha_1 d\alpha_2.$$

The mean values over  $\alpha_1$  and  $\alpha_2$  in the third term of the right hand side in (3.6) count the number of solutions of a system of equations:

$$(3.7) \quad \sum_{1 \leq k \leq s} a_k (x_k^d - z_k^d) = \sum_{1 \leq k \leq s} a_k (y_k^d - w_k^d) = 0$$

in integers  $a_k, x_k, y_k, z_k, w_k$  with  $0 < |a_k| \leq A$  and  $2^{-i}X < |x_k|, |z_k| \leq 2^{-i+1}X$  and  $2^{-j}X < |y_k|, |w_k| \leq 2^{-j+1}X$ . Meanwhile, the system (3.7) is equivalent to

$$(3.8) \quad \sum_{1 \leq k \leq s} a_k ((2^{i-1} \cdot x_k)^d - (2^{i-1} \cdot z_k)^d) = \sum_{1 \leq k \leq s} a_k ((2^{j-1} \cdot y_k)^d - (2^{j-1} \cdot w_k)^d) = 0.$$

Furthermore, one sees obviously that the number of solutions satisfying (3.8), with  $0 < |a_k| \leq A$  and  $2^{-i}X < |x_k|, |z_k| \leq 2^{-i+1}X$  and  $2^{-j}X < |y_k|, |w_k| \leq 2^{-j+1}X$ , is bounded above by

$$(3.9) \quad \sum_{1 \leq k \leq s} a_k (x_k^d - z_k^d) = \sum_{1 \leq k \leq s} a_k (y_k^d - w_k^d) = 0,$$

in integers  $a_i, x_i, y_i, z_i, w_i$  with  $0 < |a_i| \leq A$  and  $X/2 < |x_k|, |y_k|, |z_k|, |w_k| \leq X$ . This shows that

$$\int_0^1 \int_0^1 |\widetilde{W}_{ij}(\alpha_1, \alpha_2)|^s d\alpha_1 d\alpha_2 \leq \int_0^1 \int_0^1 |\widetilde{W}_{11}(\alpha_1, \alpha_2)|^s d\alpha_1 d\alpha_2,$$

for all  $1 \leq i, j \leq L$ , and hence it follows from (3.6) that

$$(3.10) \quad N(A, X) \ll X^{4s} + A^s X^{2s} + X^\epsilon \int_0^1 \int_0^1 |\widetilde{W}_{11}(\alpha_1, \alpha_2)|^s d\alpha_1 d\alpha_2.$$

On noting that the third term in (3.10) is bounded by the number of solutions satisfying (3.3), it suffices to show that the number of solutions satisfying (3.3) is bounded above by the right hand side of (3.2), in order to complete the proof of Lemma 3.1.

Let  $N_1(A, X)$  be the number of solutions counted by (3.3) where vectors  $(x_1^d - z_1^d, \dots, x_s^d - z_s^d) \in \mathbb{R}^s$  and  $(y_1^d - w_1^d, \dots, y_s^d - w_s^d) \in \mathbb{R}^s$  are linearly dependent over  $\mathbb{R}$ . First we bound the number of solutions counted by  $N_1(A, X)$  satisfying  $x_i^d - z_i^d \neq 0$  for all  $1 \leq i \leq s$ .

The linearly dependent condition implies that there exists  $c \in \mathbb{Z}$  such that

$$(3.11) \quad c(x_i^d - z_i^d) = (y_i^d - w_i^d) \text{ with } 1 \leq i \leq s.$$

Consider the case  $c = 0$ . This gives  $y_i^d = w_i^d$  for all  $1 \leq i \leq s$ . Hence, the system (3.3) of equations, satisfying (3.11) with  $c = 0$ , implies that

$$(3.12) \quad \sum_{i=1}^s a_i(x_i^d - z_i^d) = 0 \text{ and } y_i^d = w_i^d \text{ (} 1 \leq i \leq s \text{)}.$$

Therefore, since the number of solutions  $y_i$  and  $w_i$  satisfying  $y_i^d = w_i^d$  with  $|y_i|, |w_i| \leq X$  is  $O(X^s)$ , it follows by Lemma 2.2 that the number of solutions  $x_i, y_i, z_i, w_i$  satisfying (3.12) with  $0 < |a_i| \leq A$  and  $X/2 \leq |x_i|, |y_i|, |z_i|, |w_i| \leq X$  is  $O(A^s X^{2s} + A^{s-1} X^{3s-d+\epsilon})$ .

Consider the case  $c \neq 0$ . Then, the system (3.11) of equations implies

$$(3.13) \quad (y_1^d - w_1^d)(x_i^d - z_i^d) = (y_i^d - w_i^d)(x_1^d - z_1^d) \text{ (} 2 \leq i \leq s \text{)}.$$

Furthermore, the system (3.3) of equations with  $x_i^d - z_i^d \neq 0$  for all  $1 \leq i \leq s$  implies that

$$(3.14) \quad \sum_{i=1}^s a_i(x_i^d - z_i^d) = 0 \text{ and } x_i^d - z_i^d \neq 0 \text{ (} 1 \leq i \leq s \text{)}.$$

On noting that  $y_i^d - w_i^d = (y_i - w_i)(y_i^{d-1} + \dots + w_i^{d-1})$ , we infer that for given  $y_1, w_1$  and  $a_i, x_i, z_i$  ( $1 \leq i \leq s$ ) satisfying (3.14), the number of solutions  $y_i, w_i$  ( $2 \leq i \leq s$ ) is  $O(X^\epsilon)$  by the standard divisor estimate. Indeed, we ruled out the case  $y_1 = w_1$ , since otherwise  $x_1^d = z_1^d$  by (3.13) and the first equation in (3.11), which contradicts  $x_i^d - z_i^d \neq 0$  ( $1 \leq i \leq s$ ). Therefore, on noting by Lemma 2.2 that the number of solutions  $a_i, x_i, z_i$  ( $1 \leq i \leq s$ ) satisfying (3.14) is  $O(A^s X^s + A^{s-1} X^{2s-d+\epsilon})$ , and that the number of possible choices of  $y_1, w_1$  is  $O(X^2)$ , we find by discussion above that the number of solutions  $a_i, x_i, y_i, z_i, w_i$  ( $1 \leq i \leq s$ ) satisfying (3.13) and (3.14) is  $O(A^s X^{s+2} + A^{s-1} X^{2s-d+2+\epsilon})$ . To sum up, we conclude that the number of solutions counted by  $N_1(A, X)$  with  $x_1^d - z_1^d \neq 0$  for all  $1 \leq i \leq s$  is

$$(3.15) \quad O(A^s X^{2s} + A^{s-1} X^{3s-d+\epsilon}).$$

Next, assume that the number of indices  $i$  such that  $x_i^d - z_i^d = 0$  with  $1 \leq i \leq s$  is  $r$  ( $\neq 0$ ). We denote the set of these indices  $i$  by  $I$  with  $|I| = r$ . Then, the systems (3.3) and (3.11) of equations reduce to

$$(3.16) \quad \sum_{i \notin I} a_i (x_i^d - z_i^d) = \sum_{i \notin I} a_i (y_i^d - w_i^d) = 0$$

and

$$(3.17) \quad c(x_i^d - z_i^d) = y_i^d - z_i^d \quad (i \notin I)$$

and

$$(3.18) \quad y_i^d - w_i^d = 0 \quad (i \in I).$$

From the discussion leading to (3.15), we find that the number of solutions  $a_i, x_i, y_i, z_i, w_i$  ( $i \in I$ ) satisfying (3.16) and (3.17)

$$(3.19) \quad O(A^{s-r} X^{2(s-r)} + A^{s-r-1} X^{3(s-r)-d+\epsilon}).$$

Since the number of possible choices of  $a_i$  ( $i \in I$ ) is  $O(A^r)$ , and the number of solutions  $x_i, y_i, w_i, z_i$  ( $i \in I$ ) satisfying (3.18) and  $x_i^d - z_i^d = 0$  ( $i \in I$ ) is  $O(X^{2r})$ , we conclude by (3.19) that the number of solutions  $a_i, x_i, y_i, z_i, w_i$  satisfying (3.16), (3.17) and (3.18) together with  $x_i^d - z_i^d = 0$  ( $i \in I$ ) is  $O(A^s X^{2s+\epsilon} + A^{s-1} X^{3s-r-d+\epsilon})$ . Therefore, by summing over the cases with  $0 \leq r \leq s$ , we conclude that is

$$(3.20) \quad N_1(A, X) \ll A^s X^{2s} + A^{s-1} X^{3s-d+\epsilon}.$$

We denote by  $N_2(A, X)$  the number of solutions counted by (3.3) where the vectors  $(x_1^d - z_1^d, \dots, x_s^d - z_s^d) \in \mathbb{R}^s$  and  $(y_1^d - w_1^d, \dots, y_s^d - w_s^d) \in \mathbb{R}^s$  are linearly independent over  $\mathbb{R}$ . We bound this quantity via an application of the geometry of numbers. For fixed  $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$  contributing to  $N_2(A, X)$  we denote

$$\Delta_{i,j} = \det \begin{pmatrix} x_i^d - z_i^d & y_i^d - w_i^d \\ x_j^d - z_j^d & y_j^d - w_j^d \end{pmatrix}.$$

Then, just as in [5, Lemma 2.5] we have by Lemma 2.1 the following bound.

$$(3.21) \quad N_2(A, X) \ll A^{s-2} \sum_{\substack{X/2 \leq \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \leq X \\ \Delta_{i,j} \neq 0 \\ \text{for some } 1 \leq i < j \leq s}} \frac{\gcd_{1 \leq i < j \leq s} \Delta_{i,j}}{\max_{1 \leq i < j \leq s} \Delta_{i,j}}$$

$$(3.22) \quad \ll A^{s-2} \sum_{\substack{X/2 \leq x_1, y_1, z_1, w_1 \leq X \\ X/2 \leq x_2, y_2, z_2, w_2 \leq X \\ \Delta_{1,2} \geq 1}} \frac{1}{\Delta_{1,2}} \sum_{D | \Delta_{1,2}} D \psi_s(X, D),$$

where  $\psi_s(X, D) = \psi_s(X, D; x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2)$  denotes the number of solutions

$$x_3, \dots, x_s, y_3, \dots, y_s, z_3, \dots, z_s, w_3, \dots, w_s \in \mathbb{N} \cap [X/2, X],$$

to the system of congruences

$$(x_i^d - z_i^d)(y_j^d - w_j^d) \equiv (x_j^d - z_j^d)(y_i^d - w_i^d) \pmod{D} \quad \text{for all } 1 \leq i < j \leq s.$$

We prove a bound on  $\psi_s(X, D)$  when  $1 \leq D \leq 8X^{2d}$  by defining the quantity

$$m(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) = \min_{3 \leq i, j \leq s} \{(D, x_i^d - z_i^d), (D, y_j^d - w_j^d)\},$$

and consider those solutions counted by  $\psi_s(X, D)$  satisfying  $m(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) = l$ , we denote the quantity of these solutions by  $\psi_{s,l}(X, D)$ . Then by dyadically summing over the range  $1 \leq l \leq D$  one deduces that

$$(3.23) \quad \psi_s(X, D) \ll X^\varepsilon \sup_{1 \leq L \leq D} \sum_{L/2 \leq l \leq L} \psi_{s,l}(X, D).$$

Given a fixed  $1 \leq l \leq D$  we may suppose, without loss of generality, that  $(D, x_3^d - z_3^d) = l$ . Similar to the method of Brüdern and Dietmann, we ignore most restrictions and simply bound the number of solutions satisfying the congruences

$$(x_3^d - z_3^d)(y_j^d - w_j^d) \equiv (x_j^d - z_j^d)(y_3^d - w_3^d) \pmod{D} \quad \text{for all } j \neq 3,$$

with  $m(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) = l$ .

First, let us bound the number of choices for  $x_3, \dots, x_s, z_3, \dots, z_s$ . In order for a solution to be counted in  $\psi_{s,l}(X, D)$  it must satisfy

$$x_j^d - z_j^d \equiv 0 \pmod{T_j} \quad \text{for all } 3 \leq j \leq s,$$

where  $T_j \geq l$ . Recalling [5, Lemma 2.4], we deduce that the number of choices for  $x_3, \dots, x_s, z_3, \dots, z_s$  is at most

$$(3.24) \quad X^{s-2+\varepsilon} \left(1 + Xl^{-\frac{2}{d}}\right)^{s-2}.$$

We apply the same procedure to bound the number of choices for  $y_3, w_3$ , thus the contribution from these two variables is at most

$$(3.25) \quad X^{1+\varepsilon} \left(1 + Xl^{-\frac{2}{d}}\right).$$

Next, with the variables  $\mathbf{x}, y_3, \mathbf{z}, w_3$  now fixed, we bound the number of choices for  $y_4, \dots, y_s, w_4, \dots, w_s$  in two ways. Note that  $y_4, \dots, y_s, w_4, \dots, w_s$  must satisfy the following family of congruences

$$(3.26) \quad (x_3^d - z_3^d)(y_j^d - w_j^d) \equiv (x_j^d - z_j^d)(y_3^d - w_3^d) \pmod{D} \quad \text{for all } 4 \leq j \leq s.$$

By our definition of  $l$  one may deduce that there exists fixed integers  $C_j$  for each  $4 \leq j \leq s$  such that the system of congruences (3.26) is equivalent to the following system of congruences

$$(3.27) \quad y_j^d - w_j^d \equiv C_j \pmod{D/l} \quad \text{for all } 4 \leq j \leq s.$$

By considering the exponential sums which count solutions of the system (3.27) it is easily seen that the number of solutions to (3.27) is bounded above by the number of solutions to

$$y_j^d - w_j^d \equiv 0 \pmod{D/l} \text{ for all } 4 \leq j \leq s.$$

Applying [5, Lemma 2.4]  $s - 3$  times, we deduce that the number of choices for  $y_4, \dots, y_s, w_4, \dots, w_s$  is at most

$$(3.28) \quad X^{s-3+\varepsilon} \left(1 + X(D/l)^{-\frac{2}{d}}\right)^{s-3}.$$

However, one may also apply the same method we used for the variables  $\mathbf{x}, y_3, \mathbf{z}, w_3$  and obtain the bound

$$(3.29) \quad X^{s-3+\varepsilon} \left(1 + Xl^{-\frac{2}{d}}\right)^{s-3}.$$

Combining (3.24), (3.25), (3.28), (3.29) we deduce that when  $L/2 \leq l \leq L$  one has that

$$(3.30) \quad \psi_{s,l}(X; D) \ll X^{2s-4+\varepsilon} \left(1 + XL^{-\frac{2}{d}}\right)^{s-1} \left(1 + X \min\{L^{-\frac{2}{d}}, (D/L)^{-\frac{2}{d}}\}\right)^{s-3}.$$

We now list different bounds on  $\psi_{s,l}(X; D)$  by considering different cases relating the relative sizes of  $L$  to  $X^{d/2}$ ,  $D^{1/2}$ , and  $DX^{-d/2}$ .

$$\begin{aligned} X^{2s-4+\varepsilon} (XL^{-\frac{2}{d}})^{s-1}, & \quad \text{if } 1 \leq L \leq \min\{X^{\frac{d}{2}}, D^{\frac{1}{2}}, DX^{-\frac{d}{2}}\}, \\ X^{2s-4+\varepsilon} (XL^{-\frac{2}{d}})^{s-1} (X(D/L)^{-\frac{2}{d}})^{s-3}, & \quad \text{if } DX^{-\frac{d}{2}} \leq L \leq \min\{X^{\frac{d}{2}}, D^{\frac{1}{2}}\}, \\ X^{2s-4+\varepsilon} (XL^{-\frac{2}{d}})^{2s-4}, & \quad \text{if } D^{\frac{1}{2}} \leq L \leq X^{\frac{d}{2}}, \\ X^{2s-4+\varepsilon}, & \quad \text{if } X^{\frac{d}{2}} \leq L \leq D. \end{aligned}$$

Combining this with (3.23) and noting that  $2 \leq \frac{2}{d}(s-2)$  we deduce

$$(3.31) \quad \psi_s(X; D) \ll X^{4s-8+\varepsilon} D^{\frac{1}{2}-\frac{2}{d}(s-2)} + X^{3s-5+\varepsilon} + X^{2s-4+\varepsilon} D.$$

Utilizing (3.31) with (3.21), we deduce

$$N_2(A, X) \ll A^{s-2} X^\varepsilon \left( X^{2s+2d+4} + X^{3s+3} + X^{4s-8} \Xi_d(X) \right),$$

where

$$\Xi_d(X) = \sum_{\substack{X/2 \leq x_1, y_1, z_1, w_1 \leq X \\ X/2 \leq x_2, y_2, z_2, w_2 \leq X \\ \Delta_{1,2} \geq 1}} \frac{1}{\Delta_{1,2}}.$$

We now establish that

$$(3.32) \quad \Xi_d(X) \ll X^{8-2d+\varepsilon}.$$

Since  $\Delta_{1,2} \geq 1$ , one of terms  $(x_1^d - z_1^d)(y_2^d - w_2^d)$  and  $(x_2^d - z_2^d)(y_1^d - w_1^d)$  is non-zero. Then, it suffices to consider the case  $(x_2^d - z_2^d)(y_1^d - w_1^d) \neq 0$  in the summation in (3.32). Observe that

$$(3.33) \quad \min_{\substack{X/2 \leq |x_2|, |\tilde{x}_2| \leq X \\ x_2 \neq \tilde{x}_2}} |(x_2^d - z_2^d) - (\tilde{x}_2^d - z_2^d)| = \min_{\substack{X/2 \leq |x_2|, |\tilde{x}_2| \leq X \\ x_2 \neq \tilde{x}_2}} |x_2^d - \tilde{x}_2^d| \geq B,$$

with  $B = O(X^{d-1})$ . Then, by observing that for fixed  $z_2$ , the function  $x^d - z_2^d$  is monotonic increasing function in  $x$ , we infer that

$$\begin{aligned} \sum_{\substack{x_1, x_2 \\ y_1, y_2 \\ z_1, z_2 \\ w_1, w_2 \\ \Delta_{1,2} \geq 1}} \frac{1}{\Delta_{1,2}} &= \sum_{\substack{x_1, z_1 \\ y_2, w_2}} \sum_{z_2, y_1, w_1} \sum_{\substack{x_2 \\ \Delta_{1,2} \geq 1}} \frac{1}{|(x_1^d - z_1^d)(y_2^d - w_2^d) - (x_2^d - z_2^d)(y_1^d - w_1^d)|} \\ &\ll \sum_{\substack{x_1, z_1 \\ y_2, w_2}} \sum_{z_2, y_1, w_1} \sum_{1 \leq n \leq X} \frac{1}{1 + nB(y_1^d - w_1^d)} \\ &\ll X^5 \sum_{y_1, w_1} \sum_{1 \leq n \leq X} \frac{1}{1 + nB(y_1^d - w_1^d)}. \end{aligned}$$

Similarly, the last expression is seen to be

$$\begin{aligned} &X^5 \sum_{y_1, w_1} \sum_{1 \leq n \leq X} \frac{1}{1 + nB(y_1^d - w_1^d)} \\ &\ll X^5 \sum_{1 \leq n \leq X} \sum_{w_1} \sum_{1 \leq m \leq X} \frac{1}{1 + nmB^2} \\ &\ll X^6 \sum_{1 \leq n, m \leq X} \frac{1}{1 + nmB^2}. \end{aligned}$$

It follows by the standard divisor estimate that the bound in the last expression is

$$\ll X^{6+\epsilon} \sum_{1 \leq n \leq X^2} \frac{1}{1 + nB^2} \ll X^{6+\epsilon} B^{-2}.$$

Since  $B = O(X^{d-1})$ , we confirm the claim (3.32). We conclude that

$$(3.34) \quad N_2(A, X) \ll A^{s-2} X^{4s-2d+\epsilon} \left(1 + X^{2d-s+3} + X^{4d-2s+4}\right)$$

Combining (3.20) with (3.34) we deduce the claimed bound.  $\square$

#### 4. BOUNDS FOR LARGE MODULI

In this section, we provide bounds for large moduli via mean values of exponential sums that coincide with the counting problem in Lemma 3.1.

We define here the major and minor arcs. For  $B > 0$ , define the major arcs

$$(4.1) \quad \mathfrak{M}(B) = \bigcup_{\substack{0 \leq a \leq q \leq B \\ (q,a)=1}} \mathfrak{M}(q, a),$$

where

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1) : |\alpha - a/q| \leq BA^{-1}X^{-d}\},$$

and define the minor arcs  $\mathfrak{m}(B) = [0, 1) \setminus \mathfrak{M}(B)$ . Recall the definition (1.1) of  $w$ . Here and throughout, we abbreviate  $\mathfrak{M}(w)$  and  $\mathfrak{m}(w)$  simply to  $\mathfrak{M}$  and  $\mathfrak{m}$ .

Recall the definition of permutation  $[\cdot]$  following Definition 1.1. Also, we recall the definition (2.2) of  $\mathcal{I}_a(X, \mathfrak{B}) = \mathcal{I}_{[(\mathbf{b}, \mathbf{c})]}(X, \mathfrak{B})$  with a given measurable set  $\mathfrak{B} \subseteq [0, 1)$ ,  $\mathbf{b} \in \mathbb{Z}^n$  and  $\mathbf{c} \in \mathbb{Z}^{N-n}$ . Recall the definition of  $N = \binom{n+d-1}{d}$ .

**Lemma 4.1.** *Let  $n = 8s + r$  with  $s, r \in \mathbb{N}$  and  $s \geq 3d$  and  $1 \leq r \leq 8$ . Suppose that  $P \in \mathbb{Z}[\mathbf{x}]$  is a non-singular form in  $N$  variables of degree  $k \geq 2$ . Then, whenever  $1 \leq X^{2d} \leq A \leq X^{s-d}$  and  $N \geq 200k(k-1)2^{k-1}$ , there exists  $\delta' > 0$  such that we have*

$$\sum_{\substack{\|\mathbf{b}\|_\infty, \|\mathbf{c}\|_\infty \leq A \\ P([\mathbf{b}, \mathbf{c}])=0}} |\mathcal{I}_{[(\mathbf{b}, \mathbf{c})]}(X, \mathfrak{m}(X^\delta))|^2 \ll A^{N-k-2-\delta'} X^{2n-2d}.$$

*Proof.* It follows by Lemma 2.3 that

$$(4.2) \quad \begin{aligned} & \sum_{\substack{\|\mathbf{b}\|_\infty, \|\mathbf{c}\|_\infty \leq A \\ P([\mathbf{b}, \mathbf{c}])=0}} |\mathcal{I}_{[(\mathbf{b}, \mathbf{c})]}(X, \mathfrak{m}(X^\delta))|^2 \\ & \ll X^n \int_{\mathfrak{m}(X^\delta)^2} G_1(\alpha_1, \alpha_2)^{1/4} d\alpha_1 d\alpha_2 \int_0^1 G_2(\beta)^{1/4} d\beta. \end{aligned}$$

Recall the definition of  $G_1(\alpha_1, \alpha_2)$ , that is

$$G_1(\alpha_1, \alpha_2) = \sum_{\substack{\|\mathbf{h}\|_\infty \leq 2A \\ \mathbf{h} \in \mathbb{Z}^N}} |F_1(\alpha_1, \alpha_2, \mathbf{h})|^2,$$

where

$$F_1(\alpha_1, \alpha_2, \mathbf{h}) = \sum_{\substack{1 \leq \mathbf{x}, \mathbf{y} \leq X \\ \mathbf{x}, \mathbf{y} \in \mathbb{Z}^n}} e(\alpha_1 \langle \mathbf{h}, \nu_{d,n}(\mathbf{x}) \rangle - \alpha_2 \langle \mathbf{h}, \nu_{d,n}(\mathbf{y}) \rangle).$$

On recalling the definition of the permutation  $[\cdot]$ , we write  $\mathbf{h} = [(\mathbf{l}, \mathbf{m})]$  with  $\mathbf{l} \in \mathbb{Z}^n$  and  $\mathbf{m} \in \mathbb{Z}^{N-n}$ . Then, by squaring out and changing the order

of summations, one deduces by the triangle inequality that

$$G_1(\alpha_1, \alpha_2) \ll A^{N-n} \sum_{\substack{1 \leq \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \leq X \\ 1 \leq \mathbf{y}^{(1)}, \mathbf{y}^{(2)} \leq X \\ \mathbf{x}^{(i)}, \mathbf{y}^{(i)} \in \mathbb{Z}^n}} \left| \sum_{\substack{\|\mathbf{l}\|_\infty \leq 2A \\ \mathbf{l} \in \mathbb{Z}^n}} e(\Psi(\alpha_1, \alpha_2, \mathbf{l}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{y}^{(1)}, \mathbf{y}^{(2)})) \right|,$$

where

$$\begin{aligned} \Psi(\alpha_1, \alpha_2, \mathbf{l}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{y}^{(1)}, \mathbf{y}^{(2)}) \\ = \alpha_1 \langle \mathbf{l}, v_d(\mathbf{x}^{(1)}) - v_d(\mathbf{x}^{(2)}) \rangle - \alpha_2 \langle \mathbf{l}, v_d(\mathbf{y}^{(1)}) - v_d(\mathbf{y}^{(2)}) \rangle. \end{aligned}$$

By applying the Cauchy-Schwarz inequality and the triangle inequality, one sees that

$$(4.3) \quad G_1(\alpha_1, \alpha_2) \ll A^{N-n} (X^{4n})^{1/2} (A^n H(\alpha_1, \alpha_2))^{1/2},$$

where

$$H(\alpha_1, \alpha_2) = \sum_{\mathbf{l}} \left| \sum_{\substack{\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \\ \mathbf{y}^{(1)}, \mathbf{y}^{(2)}}} e(\Psi(\alpha_1, \alpha_2, \mathbf{l}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{y}^{(1)}, \mathbf{y}^{(2)})) \right|.$$

Note that

$$(4.4) \quad H(\alpha_1, \alpha_2) = \left( \sum_{-2A \leq l \leq 2A} \left| \sum_{1 \leq x \leq X} e(\alpha_1 l x^d) \right|^2 \left| \sum_{1 \leq y \leq X} e(\alpha_2 l y^d) \right|^2 \right)^n.$$

For simplicity, we temporarily write  $H(\alpha_1, \alpha_2) = W(\alpha_1, \alpha_2)^n$ , where

$$W(\alpha_1, \alpha_2) = \sum_{-2A \leq l \leq 2A} \left| \sum_{1 \leq x \leq X} e(\alpha_1 l x^d) \right|^2 \left| \sum_{1 \leq y \leq X} e(\alpha_2 l y^d) \right|^2.$$

By substituting (4.4) into (4.3) and that into (4.2), we deduce that

$$(4.5) \quad \begin{aligned} & \sum_{\substack{\|\mathbf{b}\|_\infty, \|\mathbf{c}\|_\infty \leq A \\ P([\mathbf{b}, \mathbf{c}])=0}} |\mathcal{I}_{([\mathbf{b}, \mathbf{c}])}(X, \mathbf{m}(X^\delta))|^2 \\ & \ll A^{N/4-n/8} X^{3n/2} \int_{\mathbf{m}(X^\delta)^2} H(\alpha_1, \alpha_2)^{1/8} d\alpha_1 d\alpha_2 \cdot \int_0^1 G_2(\beta)^{1/4} d\beta \\ & \ll A^{N-k-n/8} X^{3n/2} \int_{\mathbf{m}(X^\delta)^2} H(\alpha_1, \alpha_2)^{1/8} d\alpha_1 d\alpha_2, \end{aligned}$$

where we have used [13, Lemma 2.10] with  $l = 1, B = 0, A_1 = A_2 = 2A$  and  $\sigma = 1/4$  that whenever  $N \geq 200k(k-1)2^{k-1}$ , one has

$$\int_0^1 G_2(\beta)^{1/4} d\beta \ll A^{3N/4-k}.$$

Furthermore, it follows by applying the Hölder's inequality that the bound in the last expression in (4.5) is

$$(4.6) \quad \ll A^{N-k-n/8} X^{3n/2} \cdot \int_{[0,1]^2} W(\alpha_1, \alpha_2)^s d\alpha_1 d\alpha_2 \cdot \sup_{(\alpha_1, \alpha_2) \in \mathfrak{m}(X^\delta)^2} |W(\alpha_1, \alpha_2)|^{r/8}.$$

By the trivial estimate  $|\sum_{1 \leq y \leq X} e(\alpha_2 l y^d)|^2 \ll X^2$ , we deduce by [5, Lemma 4.3] that

$$(4.7) \quad \sup_{(\alpha_1, \alpha_2) \in \mathfrak{m}(X^\delta)^2} W(\alpha_1, \alpha_2) \ll AX^{4-\eta},$$

with  $\eta > 0$ . Furthermore, since the mean value  $\int_{[0,1]^2} W(\alpha_1, \alpha_2)^s d\alpha_1 d\alpha_2$  is bounded above by the number of integer solutions with  $|a_i| \leq 2A$  and  $|x_i|, |y_i|, |z_i|, |w_i| \leq X$  satisfying (3.1), we find by Lemma 3.1 that whenever  $1 \leq X^{2d} \leq A \leq X^{s-d}$  one has

$$(4.8) \quad \int_{[0,1]^2} W(\alpha_1, \alpha_2)^s d\alpha_1 d\alpha_2 \ll A^{s-2+\epsilon} X^{4s-2d}.$$

On substituting (4.7) and (4.8) into (4.6), we conclude by (4.5) that

$$\sum_{\substack{\|\mathbf{b}\|_\infty, \|\mathbf{c}\|_\infty \leq A \\ P(\mathbf{b}, \mathbf{c})=0}} |\mathcal{I}_{[(\mathbf{b}, \mathbf{c})]}(X, \mathfrak{m}(X^\delta))|^2 \ll A^{N-k-2} X^{2n-2d-\delta'},$$

for some  $\delta' > 0$ . Since  $A \leq X^{n-d}$ , we complete the proof of Lemma 4.1.  $\square$

## 5. BOUNDS FOR SMALL MODULI

In this section, we provide bounds for small moduli using the same argument as in [13, Lemma 3.8]. However, for the success of the argument with the relaxed bound on the number of variables  $n$ , we slightly modify the argument.

**Lemma 5.1.** *With the same condition in Lemma 4.1, whenever  $1 \leq X^{2d} \leq A \leq X^{s-d}$  and  $N \geq 200k(k-1)2^{k-1}$ , there exists  $\delta' > 0$  such that we have*

$$\sum_{\substack{\|\mathbf{b}\|_\infty, \|\mathbf{c}\|_\infty \leq A \\ P(\mathbf{b}, \mathbf{c})=0}} |\mathcal{I}_{[(\mathbf{b}, \mathbf{c})]}(X, \mathfrak{M}(X^\delta) \setminus \mathfrak{M}(\log X))|^2 \ll A^{N-k-2} X^{2n-2d} (\log A)^{-\eta},$$

for some  $\eta > 0$ .

*Proof.* On observing that

$$\begin{aligned}
& \mathfrak{M}(X^\delta) \setminus \mathfrak{M}(\log X) \\
&= (\mathfrak{M}(X^\delta) \setminus \mathfrak{M}((\log X)^{9d})) \cup (\mathfrak{M}((\log X)^{9d}) \setminus \mathfrak{M}(\log X)) \\
&= \left( \bigcup_{j=0}^{J_1} (\mathfrak{M}(2^{j+1}(\log X)^{9d}) \setminus \mathfrak{M}(2^j(\log X)^{9d})) \right) \\
&\quad \cup \left( \bigcup_{j=0}^{J_2} (\mathfrak{M}(2^{j+1} \log X) \setminus \mathfrak{M}(2^j \log X)) \right)
\end{aligned}$$

with  $J_1 = O(\log X)$  and  $J_2 = O(\log \log X)$ , we deduce by the Cauchy-Schwarz inequality that

$$\begin{aligned}
(5.1) \quad & \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} \left| \mathcal{I}_{\mathbf{a}}(X, \mathfrak{M}(X^\delta) \setminus \mathfrak{M}(\log X)) \right|^2 \\
& \ll J_1 \sum_{j=0}^{J_1} \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} \left| \mathcal{I}_{\mathbf{a}}(X, \mathfrak{M}(2^{j+1}(\log X)^{9d}) \setminus \mathfrak{M}(2^j(\log X)^{9d})) \right|^2 \\
& \quad + J_2 \sum_{j=0}^{J_2} \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} \left| \mathcal{I}_{\mathbf{a}}(X, \mathfrak{M}(2^{j+1} \log X) \setminus \mathfrak{M}(2^j \log X)) \right|^2.
\end{aligned}$$

Now, we analyze the mean value

$$\sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} \left| \mathcal{I}_{\mathbf{a}}(X, \mathfrak{M}(2Q) \setminus \mathfrak{M}(Q)) \right|^2,$$

with  $\log X \leq Q \leq X^\delta$ . For simplicity, we temporarily write

$$(5.2) \quad \mathfrak{C} = \mathfrak{M}(2Q) \setminus \mathfrak{M}(Q).$$

Then, by [13, Proposition 3.4], we deduce that

$$(5.3) \quad \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} \left| \mathcal{I}_{\mathbf{a}}(X, \mathfrak{C}) \right|^2 \ll A^{N-n/8-k} X^{7n/4} \int_{\mathfrak{C}} T(\alpha_1)^{\lfloor n/2 \rfloor / 8} d\alpha_1 \int_{\mathfrak{C}} T(\alpha_2)^{\lceil n/2 \rceil / 8} d\alpha_2,$$

where  $T(\alpha) = \sum_{-A \leq b \leq A} \left| \sum_{1 \leq x \leq X} e(b\alpha x^d) \right|^2$  with  $\alpha \in \mathbb{R}$ .

Meanwhile, whenever  $\alpha \in \mathfrak{C}$ , there exist  $q \in \mathbb{N}$  and  $a \in \mathbb{Z}$  with  $(q, a) = 1$  such that  $Q \leq q \leq 2Q$  and

$$|\alpha - a/q| \leq 2Q(AX^d)^{-1}.$$

Also, for  $b \in \mathbb{Z} \setminus \{0\}$  with  $|b| \leq A$ , if we write  $l = (q, b)$ ,  $\tilde{q} = q/l$  and  $\tilde{b} = b/l$ , we have

$$\left| b\alpha - \frac{a\tilde{b}}{\tilde{q}} \right| \leq \frac{2Qb}{AX^d} \leq \frac{2Q}{X^d}.$$

Hence, since  $\tilde{q} \leq 2Q$ , it follows by [12, Lemma 2.7] that

$$(5.4) \quad \sum_{1 \leq x \leq X} e(b\alpha x^d) - (\tilde{q})^{-1} S(\tilde{q}, a\tilde{b}) v(\beta) = O((2Q)^2),$$

where  $S(q, a) = \sum_{n=1}^q e(an^d/q)$  and  $v(\beta) = \int_0^X e(\beta\gamma^d) d\gamma$  with  $\beta = b\alpha - \frac{a\tilde{b}}{\tilde{q}}$ .

Thus, when  $\alpha \in \mathfrak{C}$ , we deduce from (5.4) that

$$(5.5) \quad \begin{aligned} T(\alpha) &\ll X^2 + \sum_{l|q} \sum_{\substack{|b| \leq A \\ (q,b)=l}} \left| \sum_{1 \leq x \leq X} e(b\alpha x^d) \right|^2 \\ &\ll X^2 + \sum_{l|q} \sum_{\substack{|b| \leq A \\ (q,b)=l}} \left( |((\tilde{q})^{-1} S(\tilde{q}, a\tilde{b}) v(\beta))|^2 + (2Q)^4 \right). \end{aligned}$$

By [12, Theorem 4.2], we have a bound  $S(q/l, a(b/l)) \ll (q/l)^{1-1/d}$  and a trivial bound  $v(\beta) \leq X$ . Hence, on substituting these estimates into (5.5), we find that

$$(5.6) \quad T(\alpha) \ll X^2 + \sum_{l|q} AX^2 q^{-2/d} l^{2/d-1} + AQ^4.$$

The second term  $\sum_{l|q} AX^2 q^{-2/d} l^{2/d-1}$  is bounded above by  $AX^2 q^{-2/d} \sum_{l|q} 1$ , and by the standard divisor estimate, this bound is  $O(AX^2 q^{-2/d+\epsilon})$ . Recall that  $\alpha \in \mathfrak{C}$ , and thus we have the bound  $Q \leq q \leq 2Q$  with  $\log X \leq Q \leq X^\delta$ . Furthermore, we recall the hypothesis  $X^{2d} \leq A$  in the statement in this lemma. Hence, it follows from (5.6) that

$$(5.7) \quad T(\alpha) \ll AX^2 Q^{-2/d+\epsilon}.$$

Therefore, on substituting (5.7) into (5.3), we obtain the bound

$$\sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{I}_{\mathbf{a}}(X, \mathfrak{C})|^2 \ll A^{N-n/8-k} X^{7n/4} \text{mes}(\mathfrak{C})^2 (AX^2 Q^{-2/d+\epsilon})^{n/8}.$$

On noting that  $\text{mes}(\mathfrak{C}) \ll Q^3 (AX^d)^{-1}$ , we conclude that

$$(5.8) \quad \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{I}_{\mathbf{a}}(X, \mathfrak{C})|^2 \ll A^{N-k-2} X^{2n-2d} Q^{6-n/(4d)+\epsilon}.$$

We find from (5.8) together with the hypothesis  $n = 8s + r > 24d$  in this lemma that

$$\sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{I}_\mathbf{a}(X, \mathfrak{C})|^2 \ll A^{N-k-2} X^{2n-2d} Q^{-1/(4d)+\epsilon}.$$

Hence, on recalling that  $J_1 = O(\log X)$  and  $J_2 = O(\log \log X)$ , it follows from (5.1) that

$$\sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} \left| \mathcal{I}_\mathbf{a}(X, \mathfrak{M}(X^\delta) \setminus \mathfrak{M}(\log X)) \right|^2 \ll A^{N-k-2} X^{2n-2d} (\log X)^{-\eta},$$

for some  $\eta > 0$ . On noting that  $A \leq X^{n-d}$  in the statement of this lemma, we complete the proof of Lemma 5.1.  $\square$

## 6. PROOF OF THEOREM 1.2

In this section, we provide the proof of Theorem 1.2. To do this, we require two auxiliary lemmas recorded in [13, Lemma 4.1 and 4.2]. In advance of the statement of the first lemma of these, it is convenient to define the exponential sum  $S_\mathbf{a}(q)$  with  $\mathbf{a} \in \mathbb{Z}^N$ ,  $q \in \mathbb{N}$  by

$$S_\mathbf{a}(q) := S_\mathbf{a}(q; n) = q^{-n} \sum_{\substack{1 \leq b \leq q \\ (q,b)=1}} \sum_{\substack{1 \leq \mathbf{r} \leq q \\ \mathbf{r} \in \mathbb{Z}^n}} e\left(\frac{b}{q} f_\mathbf{a}(\mathbf{r})\right).$$

**Lemma 6.1.** *Let  $n$  and  $d$  be natural numbers. Suppose that  $A, B, C$  are sufficiently large positive numbers with  $B < C$ . Suppose that  $P \in \mathbb{Z}[\mathbf{x}]$  is a non-singular form in  $N_{d,n}$  variables of degree  $k \geq 2$ . Then, for any set  $\mathcal{C} \subseteq [B, C] \cap \mathbb{Z}$ , whenever  $N \geq 200k(k-1)2^{k-1}$  we have*

$$\sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} \left| \sum_{q \in \mathcal{C}} S_\mathbf{a}(q) \right|^2 \ll A^{N-k} \left( \sum_{q \in \mathcal{C}} q^{1+\epsilon} (q^{-1} + q^{-4/d})^{n/16} \right)^2.$$

*Proof.* See [13, Lemma 4.1].  $\square$

In advance of the statement of the second auxiliary lemma, we define

$$(6.1) \quad \mathfrak{J}_\mathbf{a}(w) = X^{n-d} A^{-1} \int_{|\beta| \leq w} \int_{[0,1]^n} e(\beta A^{-1} f_\mathbf{a}(\boldsymbol{\gamma})) d\boldsymbol{\gamma} d\beta,$$

with  $\mathbf{a} \in \mathbb{Z}^N$ . Furthermore, we recall the definition (1.7) of  $\mathfrak{J}_\mathbf{a}^*$ .

**Lemma 6.2.** *Let  $n$  and  $d$  be natural numbers with  $n \geq 8(d+1)$ . Suppose that  $P \in \mathbb{Z}[\mathbf{x}]$  is a non-singular form in  $N_{d,n}$  variables of degree  $k \geq 2$ .*

Then, whenever  $N \geq 200k(k-1)2^{k-1}$ , we have

$$\sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathfrak{J}_\mathbf{a}^* - \mathfrak{J}_\mathbf{a}(w)|^2 \ll A^{N-k-2} X^{2n-2d} (w^{2-n/(2(d+1))} + \zeta).$$

*Proof.* Only difference between Lemma 6.2 and [13, Lemma 4.2] is the choice of  $\zeta$ . One readily sees that the choice of  $\zeta = w^{-4-1/(8d)}$  does not harm the argument in [13, Lemma 4.2], and thus [13, Lemma 4.2] still holds with  $\zeta = w^{-4-1/(8d)}$ .  $\square$

The proof of Theorem 1.2 follows the same strategy in the proof of [13, Theorem 1.1] with  $\zeta = w^{-4-1/(8d)}$ . In order to make this paper self-contained, we record here this proof again using  $\zeta = w^{-4-1/(8d)}$ .

*Proof of Theorem 1.2.* Recall the definition (2.2) and (4.1) of  $\mathcal{I}_\mathbf{a}(X, \mathfrak{B})$  and  $\mathfrak{M}$ . Then, we find that

$$\begin{aligned} \mathcal{I}_\mathbf{a}(X, \mathfrak{M}) &= \int_{\mathfrak{M}} \sum_{\substack{1 \leq \mathbf{x} \leq X \\ \mathbf{x} \in \mathbb{Z}^n}} e(\alpha f_\mathbf{a}(\mathbf{x})) d\alpha \\ &= \sum_{1 \leq q \leq w} \sum_{\substack{1 \leq b \leq q \\ (q,b)=1}} \int_{|\alpha - b/q| \leq \frac{w}{AX^d}} \sum_{1 \leq \mathbf{x} \leq X} e(\alpha f_\mathbf{a}(\mathbf{x})) d\alpha. \end{aligned}$$

Recall the definition (6.1) of  $\mathfrak{J}_\mathbf{a}(w)$ . By applying classical treatments in major arcs [2, Lemma 5.1] and writing  $\beta = \alpha - b/q$ , we readily find that

$$(6.2) \quad \mathcal{I}_\mathbf{a}(X, \mathfrak{M}) = \sum_{1 \leq q \leq w} S_\mathbf{a}(q) \mathfrak{J}_\mathbf{a}(w) + O(A^{-1} X^{n-d-1} w^5),$$

where

$$S_\mathbf{a}(q) = \sum_{\substack{1 \leq b \leq q \\ (q,b)=1}} q^{-n} \sum_{1 \leq \mathbf{r} \leq q} e\left(\frac{b}{q} f_\mathbf{a}(\mathbf{r})\right).$$

Meanwhile, recall the definition (1.2) and (1.5) of  $W$  and  $\mathfrak{S}_\mathbf{a}^*$ , and note from the classical treatment that

$$\mathfrak{S}_\mathbf{a}^* = \prod_{p \leq w} \left( \sum_{0 \leq h \leq \log_p w} S_\mathbf{a}(p^h) \right).$$

If we define a set

$$\mathcal{Q} = \{q \in (w, W] : \text{for all primes } p, p^h \| q \Rightarrow p^h \leq w\}$$

and define

$$\mathcal{E}_\mathbf{a} = \sum_{q \in \mathcal{Q}} S_\mathbf{a}(q),$$

we find from the multiplicativity of  $S_\mathbf{a}(q)$  that

$$(6.3) \quad \sum_{1 \leq q \leq w} S_\mathbf{a}(q) = \mathfrak{S}_\mathbf{a}^* - \mathcal{E}_\mathbf{a}.$$

Meanwhile, note that

$$\mathcal{I}_a(X) = \mathcal{I}_a(X, \mathfrak{M}) + \mathcal{I}_a(X, \mathfrak{m}).$$

Then, on recalling the definition of  $\mathfrak{J}_a^*$ , we deduce from (6.2) and (6.3) together with applications of elementary inequality  $(a+b)^2 \leq 2a^2 + 2b^2$  that

$$(6.4) \quad \begin{aligned} & \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{I}_a(X) - \mathfrak{G}_a^* \mathfrak{J}_a^*|^2 \\ & \ll \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{I}_a(X, \mathfrak{M}) - \mathfrak{G}_a^* \mathfrak{J}_a^*|^2 + \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{I}_a(X, \mathfrak{m})|^2 \\ & \ll \Sigma_1 + \Sigma_2 + \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{I}_a(X, \mathfrak{m})|^2 + O\left(A^{-2} X^{2n-2d-2} w^{10} \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} 1\right), \end{aligned}$$

where

$$\Sigma_1 = \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{E}_a \mathfrak{J}_a^*|^2 \quad \text{and} \quad \Sigma_2 = \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |(\mathfrak{G}_a^* - \mathcal{E}_a)(\mathfrak{J}_a^* - \mathfrak{J}_a(w))|^2.$$

First, we estimate the third and fourth terms of the last expression in (6.4). Since we have

$$\#\{\mathbf{a} \in \mathbb{Z}^N : \|\mathbf{a}\|_\infty \leq A, P(\mathbf{a}) = 0\} \ll A^{N-k},$$

it follows by Lemma 4.1 that

$$(6.5) \quad \begin{aligned} & \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{I}_a(X, \mathfrak{m})|^2 + O\left(A^{-2} X^{2n-2d-2} w^{10} \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} 1\right) \\ & \ll A^{N-k-2} X^{2n-2d} (\log A)^{-\delta}, \end{aligned}$$

with some  $\delta > 0$ .

Next, we turn to estimate the first term of the last expression in (6.4). From the trivial bound, we have

$$(6.6) \quad |\mathfrak{J}_a^*|^2 \ll X^{2(n-d)} \zeta^{-2} A^{-2} = X^{2(n-d)} w^{8+1/(4d)} A^{-2}.$$

By Lemma 6.1 with  $k = 2$ ,  $B = w$ ,  $C = W$  and  $\mathcal{C} = \mathcal{Q}$ , we have

$$(6.7) \quad \begin{aligned} & \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{E}_a|^2 \ll A^{N-k} \left( \sum_{q \geq w} q^{1+\epsilon} (q^{-1} + q^{-4/d})^{n/16} \right)^2 \\ & \ll A^{N-k} \left( \sum_{q \geq w} \left( q^{1-n/16+\epsilon} + q^{1-n/(4d)+\epsilon} \right) \right)^2. \end{aligned}$$

Meanwhile, from the hypotheses  $n = 8s + r$  with  $s \in \mathbb{N}$  and  $1 \leq r \leq 8$ ,  $s \geq 3d$  and  $X^{2d} \leq A \leq X^{s-d}$  in the statement of Theorem 1.2, we see that

$n > 24d$ . Hence, it follows from (6.7) together with the hypothesis  $d \geq 4$  that

$$(6.8) \quad \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{E}_\mathbf{a}|^2 \ll A^{N-k} \left( w^{2-3d/2} + w^{-4-1/(4d)} \right)^2 \ll A^{N-k} \cdot w^{-8-1/(2d)}.$$

Therefore, combining (6.6) and (6.8), we conclude that

$$(6.9) \quad \Sigma_1 = \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{E}_\mathbf{a} \mathfrak{J}_\mathbf{a}^*|^2 \ll A^{N-k-2} X^{2n-2d} w^{-1/(4d)}.$$

Lastly, it remains to estimate the second term of the last expression in (6.4). From the trivial bound, we have

$$|\mathfrak{S}_\mathbf{a}^* - \mathcal{E}_\mathbf{a}|^2 = \left| \sum_{1 \leq q \leq w} S_\mathbf{a}(q) \right|^2 \leq w^4.$$

Hence, we deduce by applying Lemma 6.2 with  $n > 24d$  and  $d \geq 4$  that

$$(6.10) \quad \Sigma_2 = \sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |(\mathfrak{S}_\mathbf{a}^* - \mathcal{E}_\mathbf{a})(\mathfrak{J}_\mathbf{a}^* - \mathfrak{J}_\mathbf{a}(w))|^2 \ll A^{N-k-2} \cdot X^{2n-2d} \cdot w^{-1/(8d)}.$$

Then, on recalling the definition of  $w$  and substituting (6.5), (6.9) and (6.10) into the last expression in (6.4), one concludes that

$$\sum_{\substack{\|\mathbf{a}\|_\infty \leq A \\ P(\mathbf{a})=0}} |\mathcal{I}_\mathbf{a}(X) - \mathfrak{S}_\mathbf{a}^* \mathfrak{J}_\mathbf{a}^*|^2 \ll A^{N-k-2} X^{2n-2d} (\log A)^{-\delta},$$

for some  $\delta$  with  $0 < \delta < 1$ . This completes the proof of Theorem 1.2.  $\square$

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*Email address:* `flore205@purdue.edu`

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, 150 N. UNIVERSITY STREET,  
WEST LAFAYETTE, IN 47907-2067

*Email address:* `kyeon@ucdavis.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, ONE SHIELDS  
AVENUE, DAVIS, CA 95616, USA